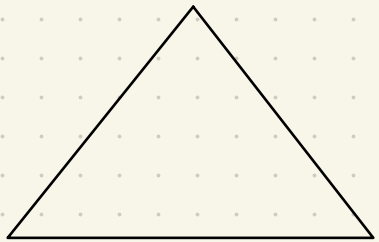
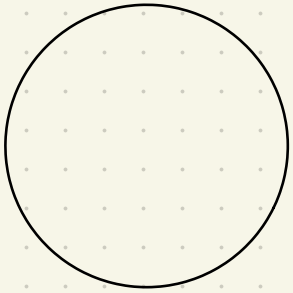
Group Theory:
Symmetry, Knots and Monsters

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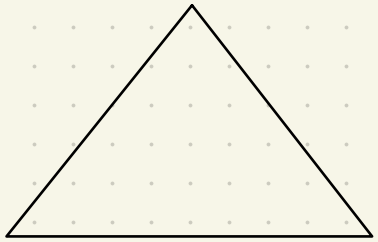
1. Symmetry

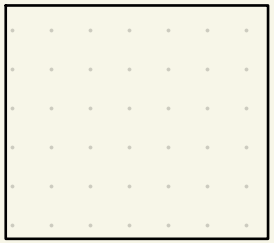
Which of the following shapes has more symmetry?





It's only slightly harder to answer the same question for the equilateral triangle compared to the square:

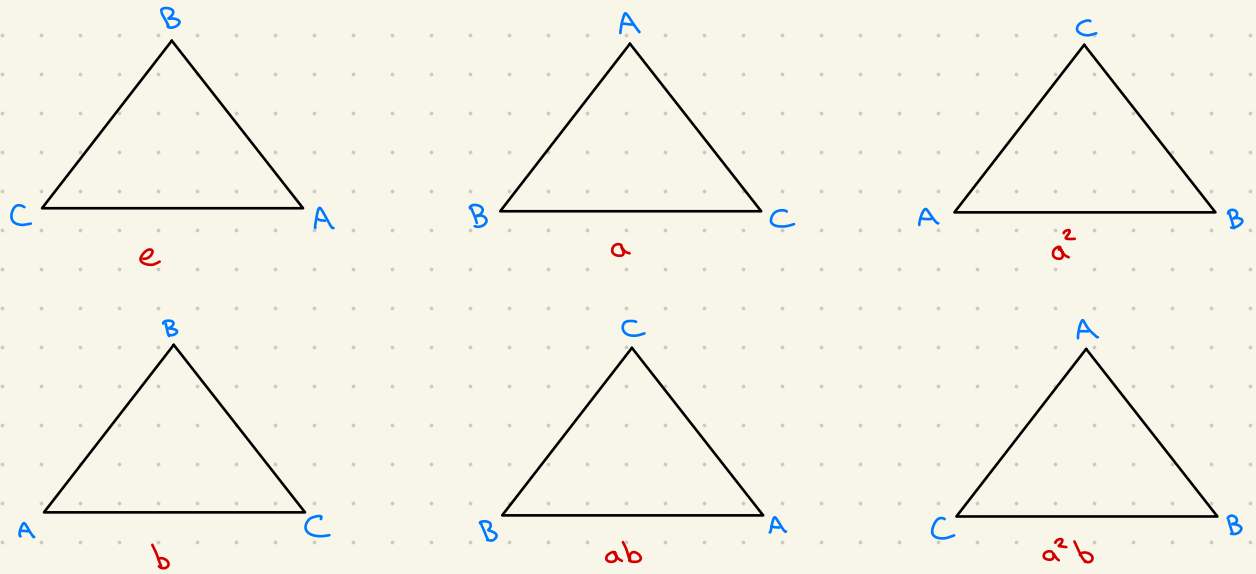


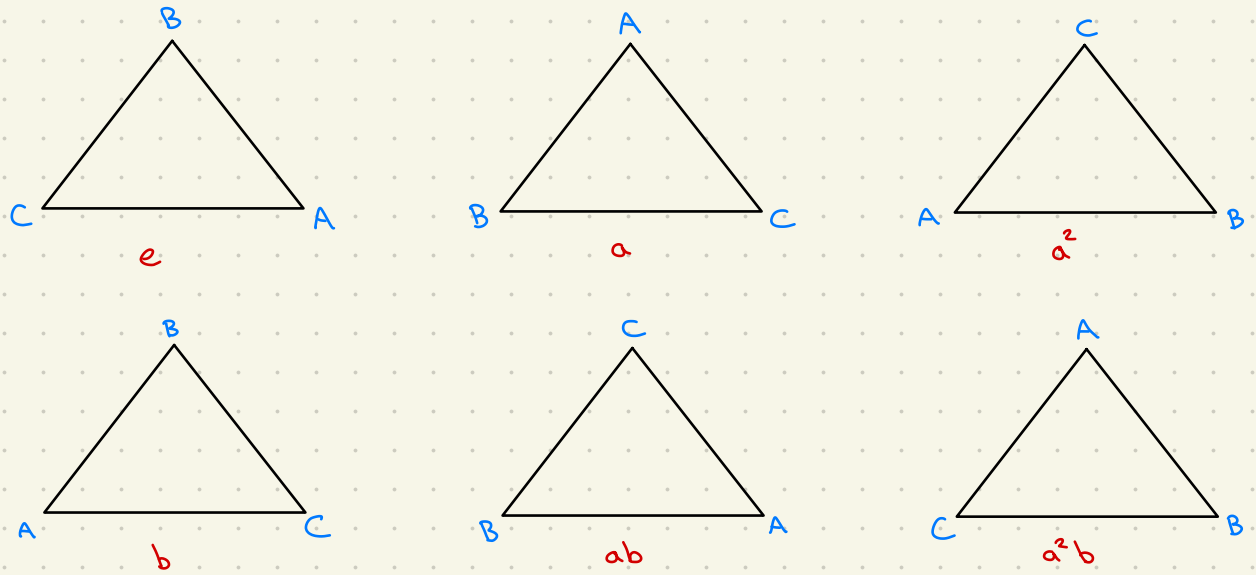


Group theory is a way to encode the symmetry transformations algebraically.

A symmetry transformation is a transformation that not only preserves the shape but also the distance between points on the polygon.

The triangle has six symmetry transformations:





Each transformation can be built out of two fundamental transformations:

a denotes a rotation by $2\pi/3$ (i.e. 120°)

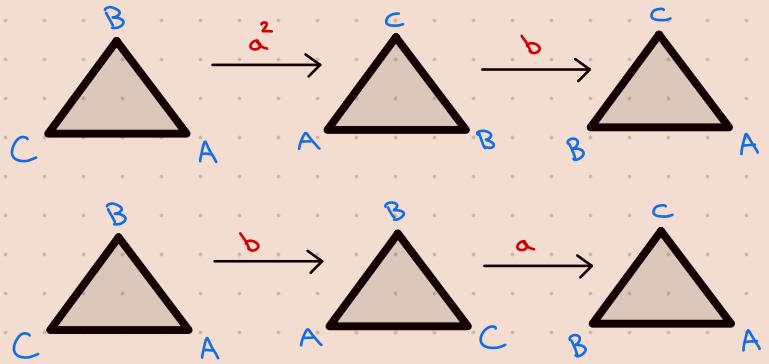
b denotes a reflection in the vertical line bisecting the triangle.

The symmetry group of the triangle is then denoted by

$\langle a, b \rangle$ with the relations $a^3 = e$, $b^2 = e$ and $ab = ba^2$

Stands for the span of a and b: it contains all the "words" you can write down using the symbols a and b. E.g. aaaba, ab, bababbbbbbb, and so on.

The relations encode what we know about the symmetries of the triangle e.g.



The infinite list of words is reduced to just six words by applying the relations.

The symmetry group of the equilateral triangle is our prototype of a group:

Definition 1.1. *A group is a set G and a mapping from the Cartesian product $G \times G$ into G (a* law of composition *or* multiplication law*), which we will denote by juxtaposition*

$$G \times G \to G \quad : \quad (g_1, g_2) \mapsto g_1 g_2, \tag{1.1}$$

with the following properties:

(i) Associativity: $g_1(g_2 g_3) = (g_1 g_2) g_3$ *for all* $g_1, g_2, g_3 \in G$

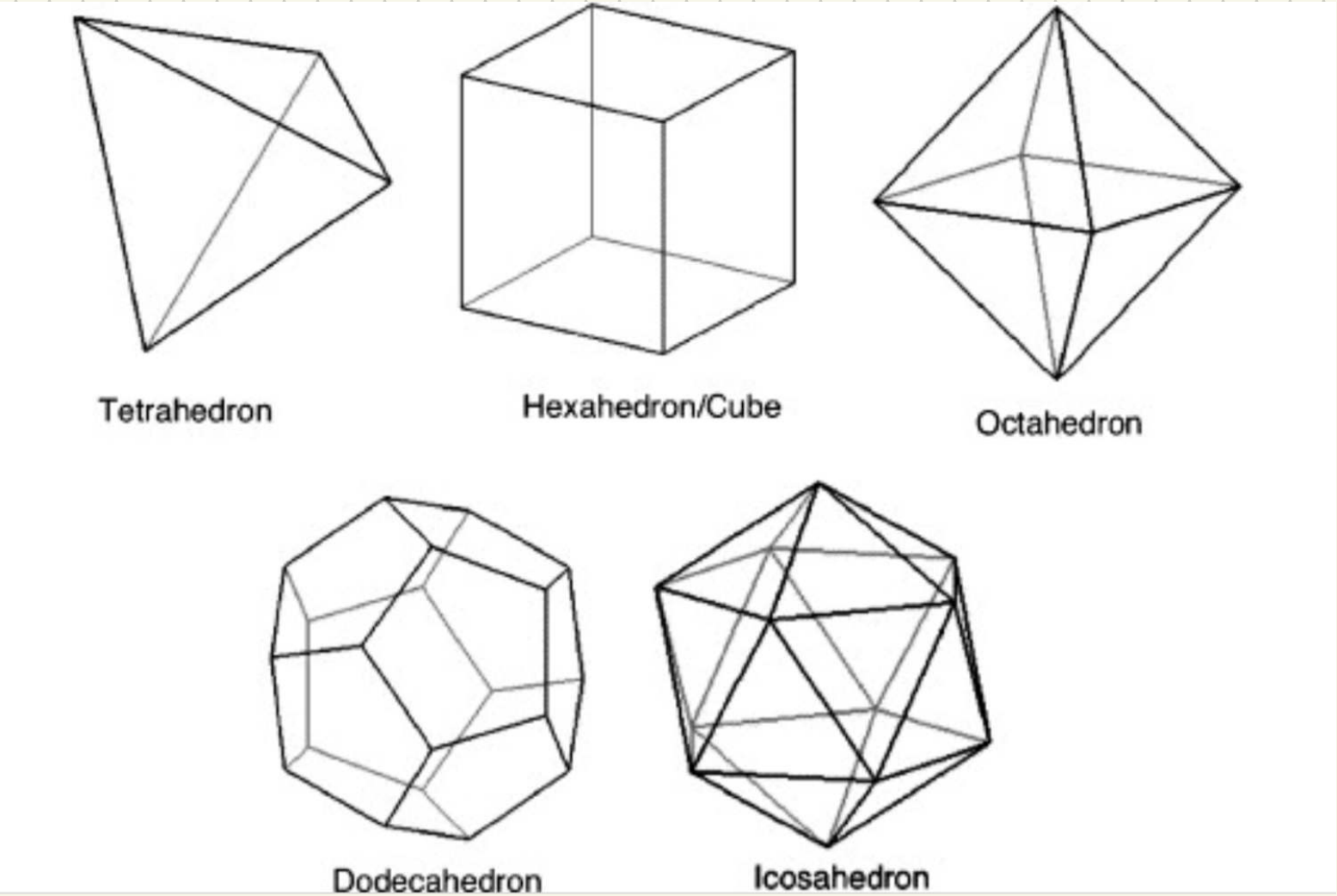
(ii) Identity: there exists $e \in G$*, called an identity, such that* $ge = eg = g$ *for all* $g \in G$

(iii) Inverse: for all $g \in G$ *there exists* $g^{-1} \in G$*, called an inverse of* g*, such that* $gg^{-1} = g^{-1}g = e$*, where* e *is an identity in* G*.*

For the symmetries of the triangle, the inverse transformations are:

$e^{-1} = e$, $a^{-1} = a^2$, $(a^2)^{-1} = a$, $b^{-1} = b$, $(ab)^{-1} = b^{-1}a^{-1} = ba^2 = ab$, $(a^2b)^{-1} = b^{-1}(a^2)^{-1} = ba = a^2b$.

The symmetries of the five Platonic solids are built out of those of the regular polygons.



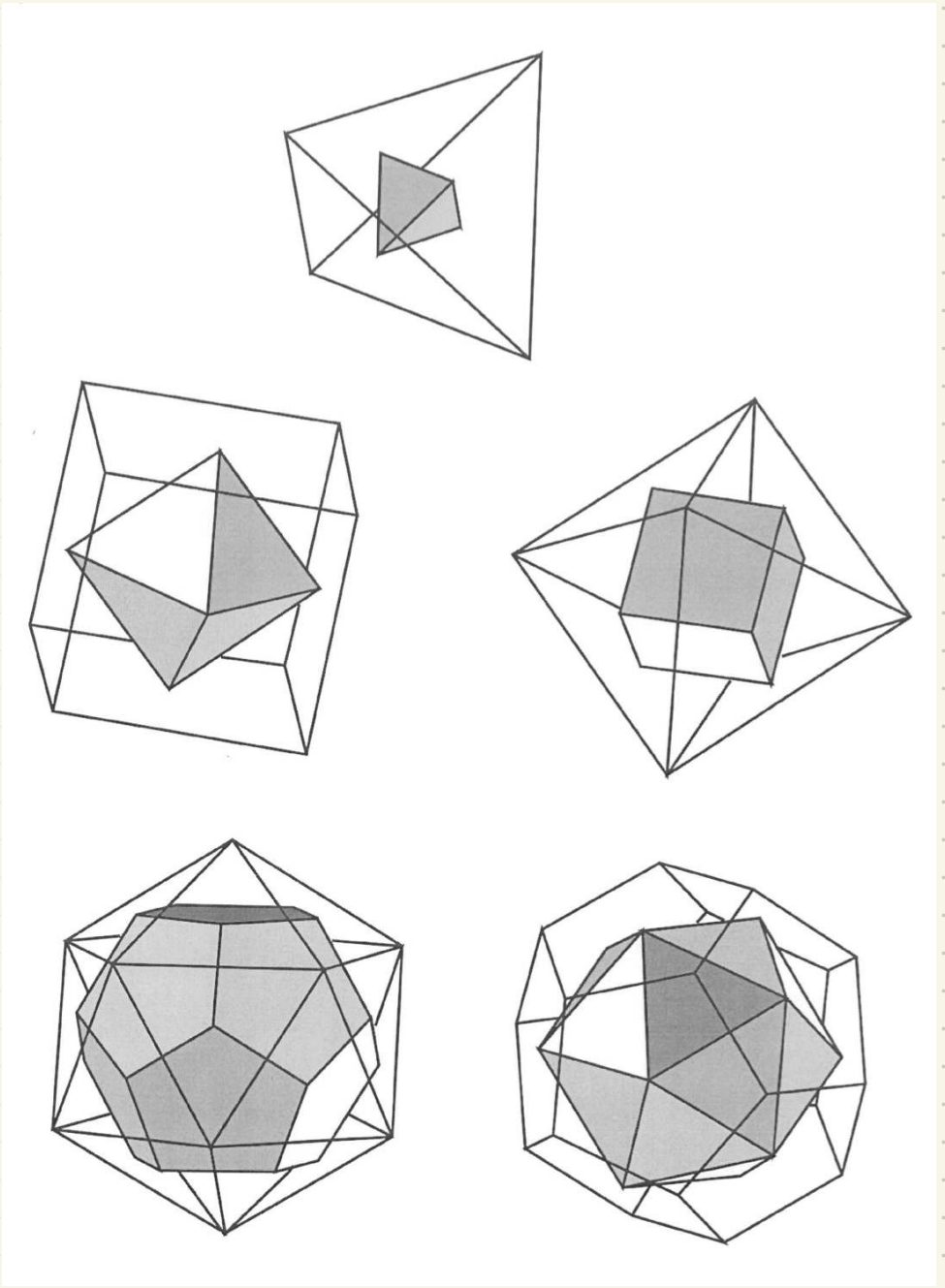
The set of Platonic solids possesses a symmetry transformation:

Swapping vertices for the centres of a face maps the shapes as

Tetrahedron <—> Tetrahedron

Cube <—> Octahedron

Dodecahedron <—> Icosahedron



Polyhedron		Vertices	Edges	Faces	Schläfli symbol	Vertex configuration
tetrahedron		4	6	4	{3, 3}	3.3.3
cube		8	12	6	{4, 3}	4.4.4
octahedron		6	12	8	{3, 4}	3.3.3.3
dodecahedron		20	30	12	{5, 3}	5.5.5
icosahedron		12	30	20	{3, 5}	3.3.3.3.3

A symmetry transformation must leave vertices unchanged, hence the vertex/edge duality tells us that:

The cube and the octahedron have the same symmetry group, as do the octahedron and the icosahedron.

There are only 3 symmetry groups for the 5 Platonic solids: the groups are more fundamental than the Platonic solids.

In addition symmetry groups may be symmetries of more abstract objects such as algebraic equations, or even theories of the universe.

The term group was coined by Évariste Galois:



Évariste Galois (1811 – 1832)



Paolo Ruffini

Paolo Ruffini (1765 – 1822).



Niels Henrik Abel (1802-1829).

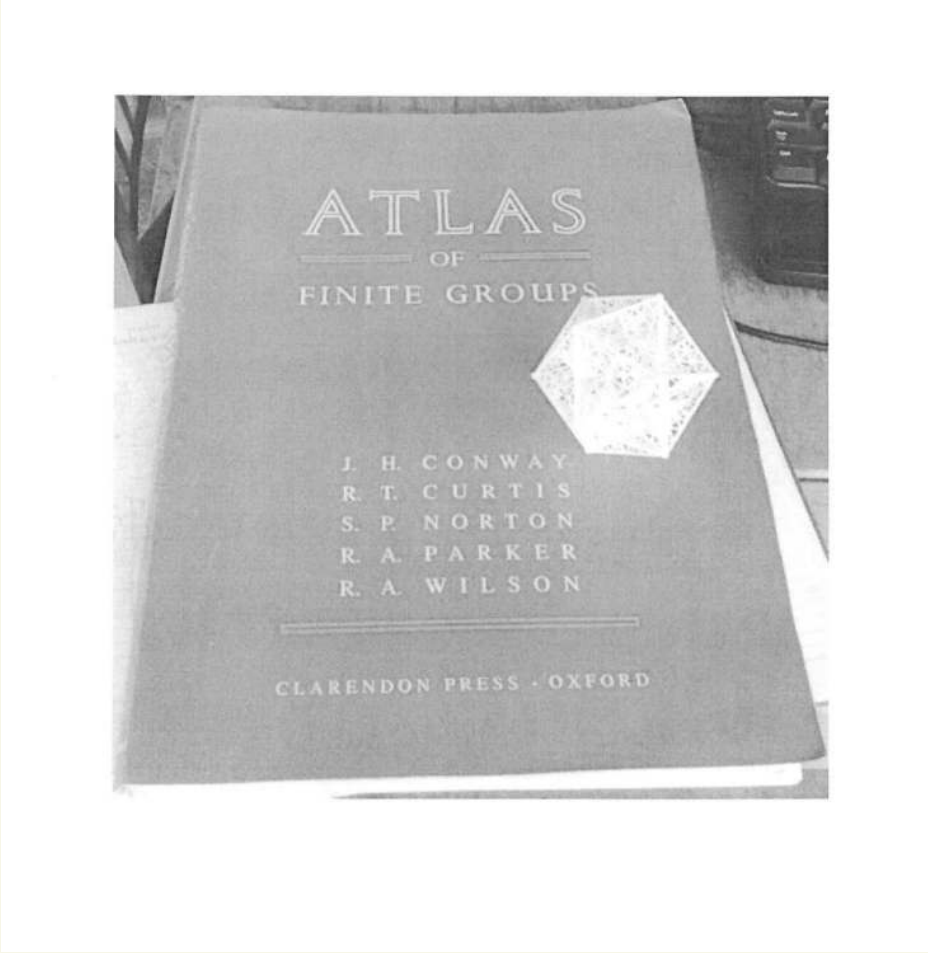


Augustin-Louis Cauchy (1789-1857)



Jean-Baptiste Joseph Fourier (1768-1830).

One of the great achievements of the 20th century was the classification of the finite simple groups in the Atlas of finite groups (published in 1983)



The order of a group G is the number of elements in the group and is denoted |G|. The order of the group of symmetries of the equilateral triangle is 6.

We can construct the groups of low order and write out their multiplication table (Cayley table):

Groups of order 1

Let G={e}

	e
e	e

Groups of order 2

Let G={e,a}

	e	a
e	e	a
a	a	a^2

As the group is of order two, then either

$$a^2 = e \quad \text{or} \quad a^2 = a$$

But if we multiply the second choice by the inverse to a we find:

$$a^{-1} a^2 = a^{-1} a$$

$$\Rightarrow a = e$$

However if a=e then we only have a group of order one.

Hence it must be that the group of order two is given by

	e	a
e	e	a
a	a	e

In terms of symbols and relations this is written:

$$\langle a \rangle \quad \text{with} \quad a^2 = e$$

It is called the cyclic group of order two and denoted $\mathbb{Z}_2$.

For larger group tables the construction is made easier once we realise that each group element can appear once and only once in each row or column.

Suppose the contrary was possible and in a row we found that

$$ab = d \quad \text{and} \quad ac = d$$

Multiply by a^{-1}.

$$b = a^{-1}d \quad \text{and} \quad c = a^{-1}d$$

$$\Rightarrow b = c$$

In which case b and c are identical and so correspond to only one column in the group multiplication table.
A similar argument holds for columns.

Groups of order 3

Let $G=\{e,a,b\}$

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

N.B. $a^2=b$ and $a^3=e$. We have $\langle a \rangle$ with $a^3=e$.

This is the cyclic group of order 3, denoted $\mathbb{Z}_3$.

Groups of order 4

Let G={e,a,b,c}

	e	a	b	c
e	e	a	b	c
a	a	e,b (or c)		
b	b			
c	c			

$a^2 = b$

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

$\mathbb{Z}_4$

$a^2 = e$

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a or e	
c	c	b		

$b^2 = a$

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

$b^2 = e$

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

swap b and c

The Klein group

$V_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

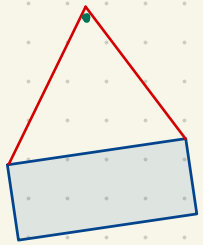
2. Knots

Groups may act or be encoded in all sorts of settings. We will think about some groups associated to knots.

The Picture-Hanging Puzzle (of A. Spivak)

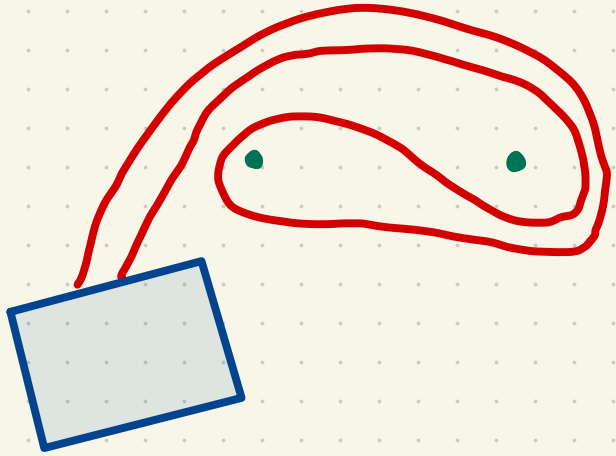
We have considered groups given by symbols and relations. What happens if a group has no extra relations beyond the group axioms? Then we have a free group.

It is risky to hang a picture by one nail: the nail could fail and the picture would fall.



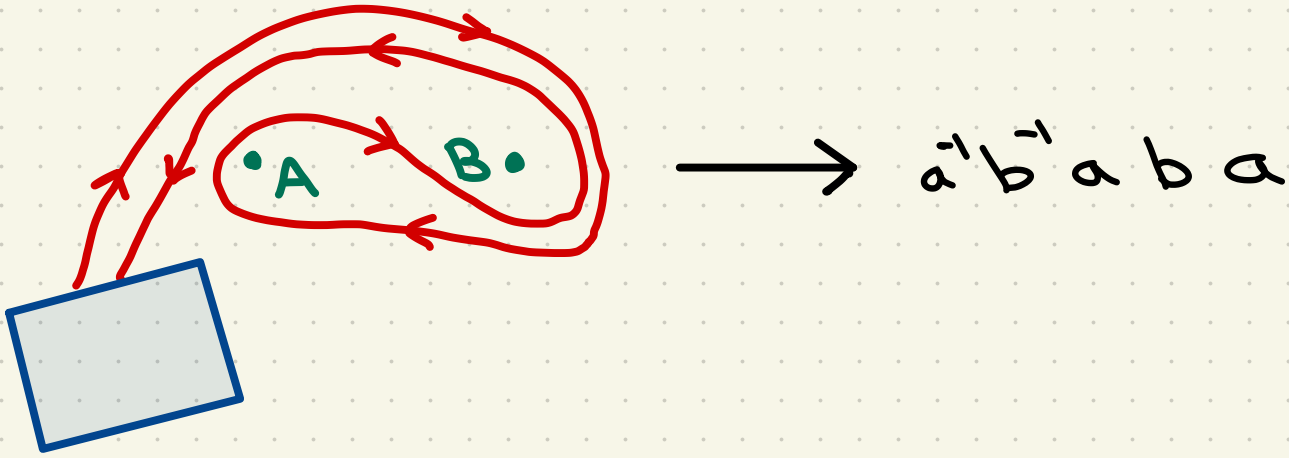
The picture-hanging puzzle asks when it is just as risky to hang a picture by two nails.

Will the tangling of the string about the two nails below unravel if either of the nails is removed?



Group theory can help us encode the way the string loops over the nails.

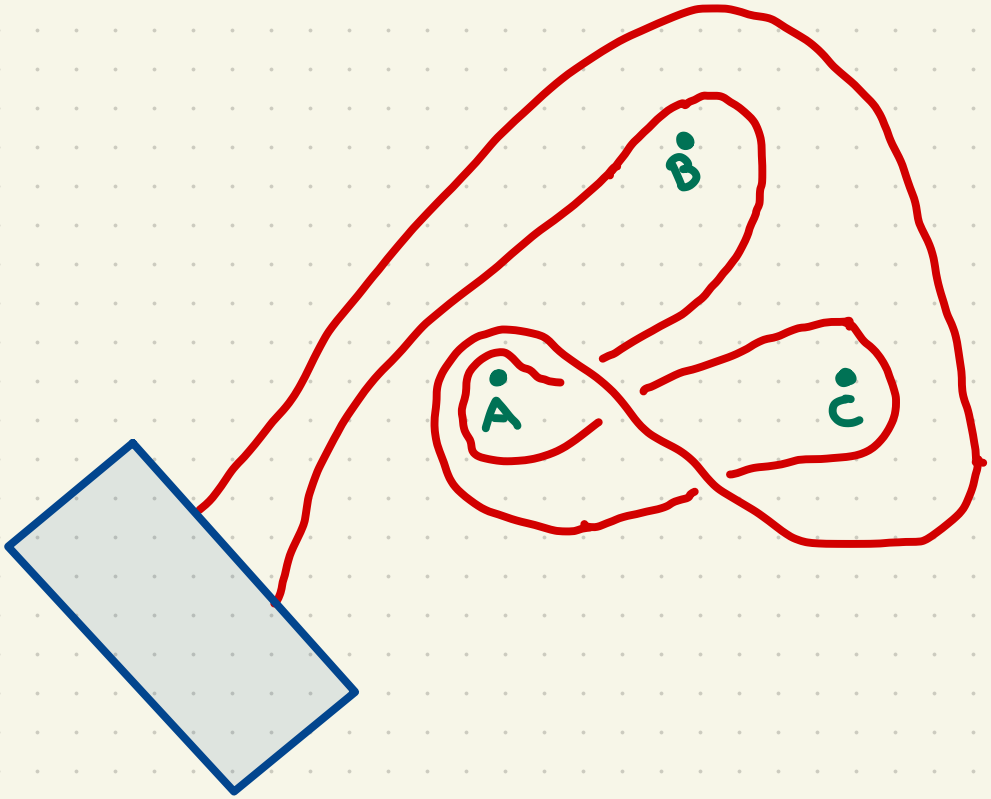
- Give the nails some names A and B.
- If the string loops clockwise over A denote that by a and if it loops clockwise over B denote that by b.
- Let anti-clockwise loops be given by the inverse elements.



If A is removed: set $a=e$
then $a^{-1}b^{-1}aba = b^{-1}b = e$
it will fall.

If B is removed set $b=e$ then
$a^{-1}b^{-1}aba = a^{-1}aa = a$
it will still loop around A.

How about?



[Hat-tip to Professors Gérard Watts and Konnie Rietsch, for more see the paper at https://arxiv.org/pdf/1203.3602.pdf]

The Braid Groups

Consider:

- two boards with n pegs on each
- n threads connect the two sets of pegs
- threads may not entangle with themselves

This construction is called an n-braid. E.g.

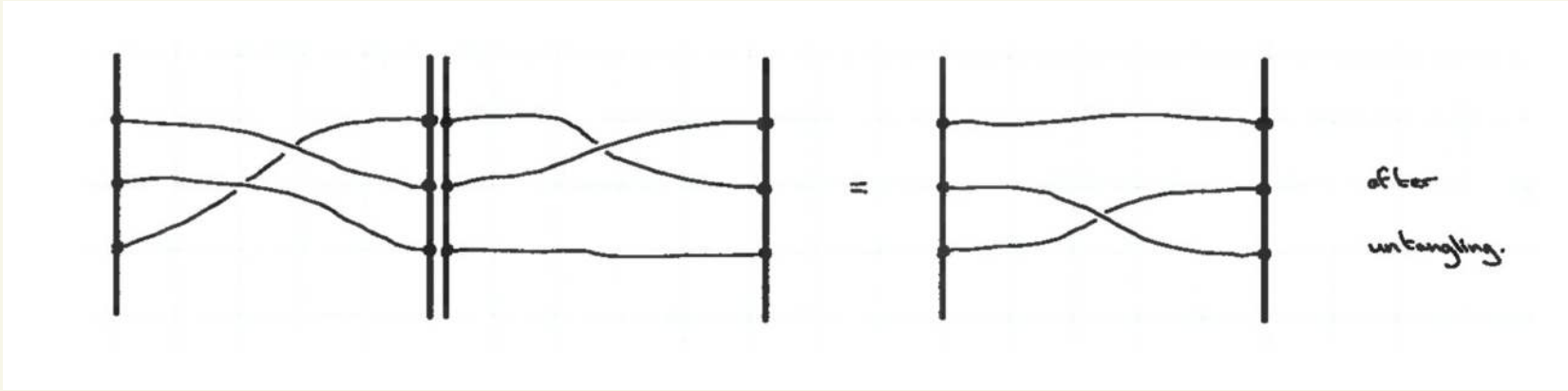


Are examples of 3-braids.

Braids can be multiplied by composing them as follows:

- Line any two braids up head-to-tail,
- connect the aligned threads and
- remove the interior boards.

E.g.

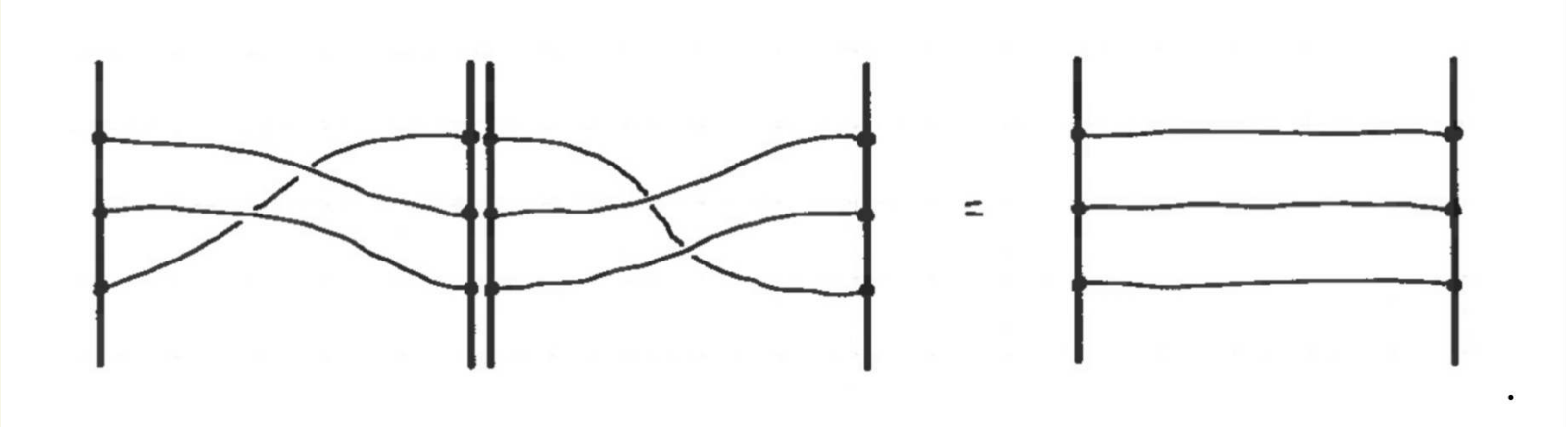


The identity 3-braid is given by:



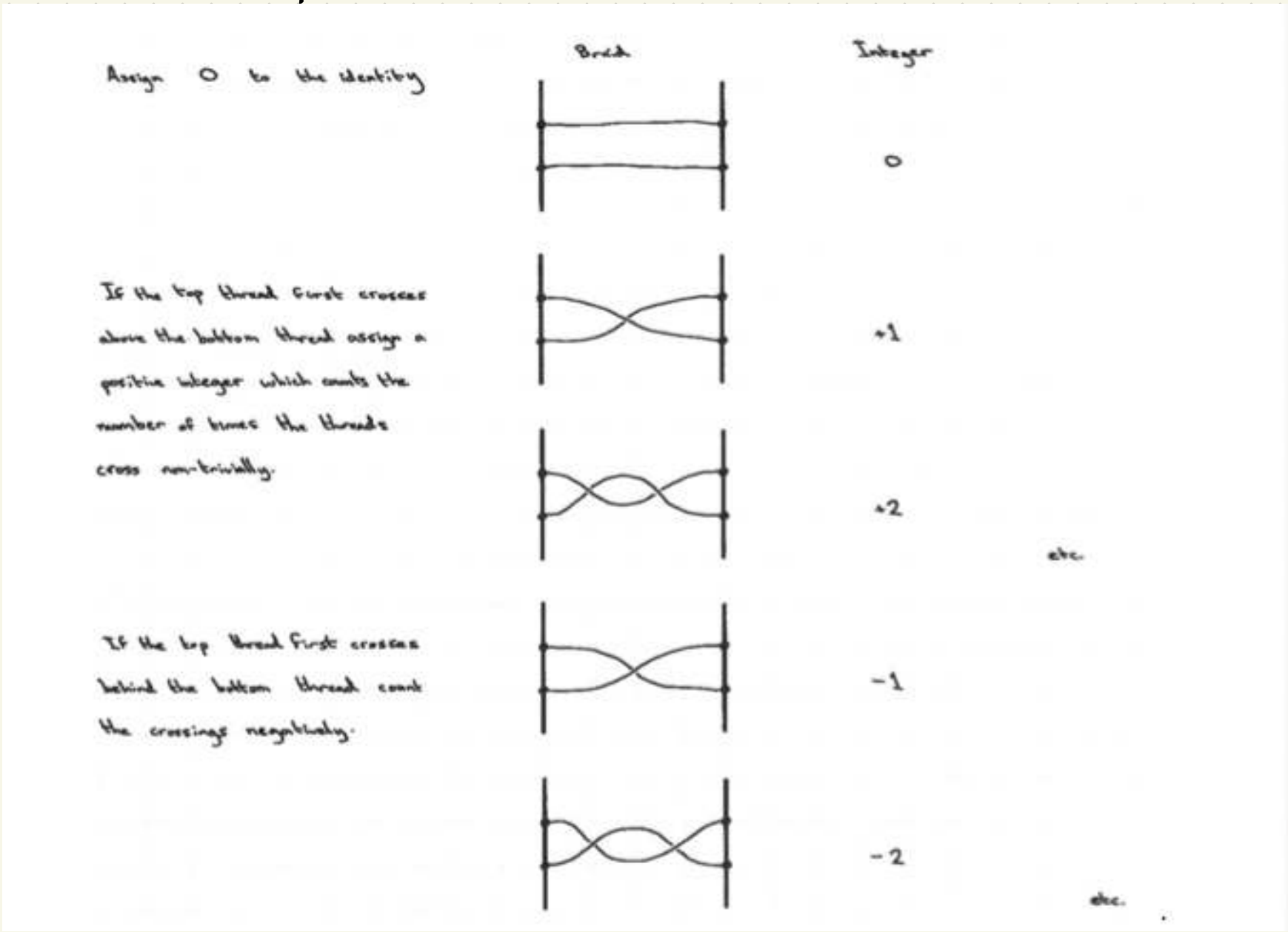
Given any 3-braid, can you construct the inverse braid?

To find the inverse to any braid you just have to rotate it around so that board two is rotated to the position of board one and vice-versa:



The n-braids form a group denoted B_n.

The 2-braid group is the integers in disguise:



What do braids have to do with knots?

The answer is that every braid can be converted into a knot or set of knots, by connecting the threads of board two up to those of board one, forming closed loops of threads, often knotted.

Braids give a nice computational way to describe knots.

Conway's Rational Tangles

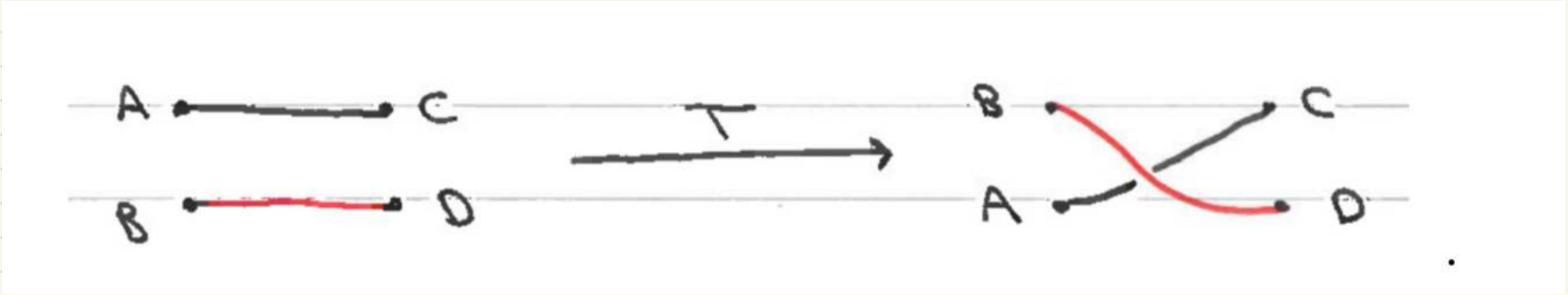
Commence with the identity 2-braid and label the corners:



We will define two simple actions on this:

1. Twist 'em up (T): B lifts their thread up and A passes underneath.

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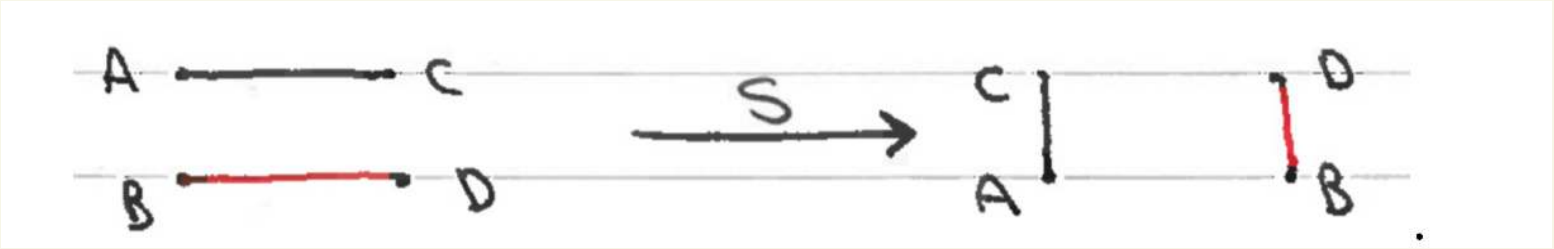


This gives the 1-tangle: there is one overlap of the threads and as we look from left to right the top-left thread (now held by B) overlaps the other string once. If it had gone under once we would call this the -1 tangle.

Algebraically,

$$T(0) = 1.$$

2. Spin 'em round (S): where A, B, C and D rotate anti-clockwise by ninety degrees.



We get an undefined tangle, what is the tangle number for this configuration? Can we give it some meaning?

$$S(0) = ?$$

Some Exercises!

1. Find an algebraic expression for $T(n)$.
2. Show that $S^2(n) = n$.
3. Untangle $T(0)$ using only S and T actions, and hence show that $S(1) = -1$.
4. Consider $T^k S(0)$ - what does this imply for $S(0)$ as a number? Can you make sense of this?
5. Undo the 2-tangle using S and T actions.
6. If we could swap all the crossings in a tangle, so that a string crossing under another becomes one that passes over, we would map an n-tangle to its negative the -n-tangle. Call this the mirror. With this in mind compare $ST^2(0) = m$ with $TST^2(0)$ to find the unknown m.

By now we may guess that:

$$T(n) = n+1 \qquad S(n) = -1/n.$$

To check these guesses, we may use them to try to untangle some complicated tangles, e.g.

$$\begin{aligned} TST^3ST^2(0) &= TST^3S(2) \\ &= TST^3(-1/2) \\ &= TS(5/2) \\ &= T(-2/5) \\ &= 3/5. \end{aligned}$$

$$3/5 \xrightarrow{S} -5/3 \xrightarrow{T^2} 1/3 \xrightarrow{S} -3 \xrightarrow{T^3} 0.$$

There is a simple relation on the symbols S and T:

$$(ST)^3 = e$$

as $$STSTST(n) = STSTS(n+1)$$
$$= STST\left(\tfrac{-1}{(n+1)}\right)$$
$$= STS\left(\tfrac{n}{n+1}\right)$$
$$= ST\left(\tfrac{-(n+1)}{n}\right)$$
$$= S\left(\tfrac{-1}{n}\right)$$
$$= n.$$

$$\therefore T^{-1} = STSTS$$

With this inverse we can guarantee that we can untangle any tangle!

But... it may not give the most efficient untangling for example consider again the 3/5- tangle given by

$$TST^3ST^2(0)$$

The inverse element is

$$\begin{aligned} T^{-2}S^{-1}T^{-3}S^{-1}T^{-1} &= (STSTS)^2S(STSTS)^3S(STSTS) \\ &= STSTSSTSTSSSTSTSSTSTSSTSTSSSTSTS \\ &= STSTTSTSTSTTSTTSTSTSTS \\ &= STSTTTSTTS \\ &= STST^3ST^2S \end{aligned}$$

Which has some redundancy as:

$$3/5 \xrightarrow{S} -5/3 \xrightarrow{T^2} 1/3 \xrightarrow{S} -3 \xrightarrow{T^3} 0 \xrightarrow{S} -\infty \xrightarrow{T} -\infty \xrightarrow{S} 0.$$

From the tangles, we have another group: it is called the modular group and is defined by

$$\langle S, T \rangle \quad \text{with} \quad S^2 = e, \quad (ST)^3 = e.$$

The modular group appears in many diverse areas of mathematics and physics, from the mock modular forms found by Ramanujan to the computation of the entropy of a black hole in string theory.

3. Monsters.

The largest of all the sporadic, finite groups is called the Monster group.

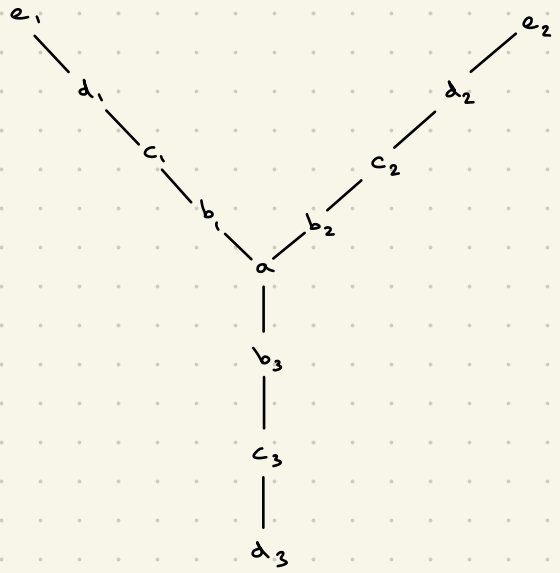
It's discovery coincided with the end of 150 years of searching to discover new symmetry groups.

The order of the Monster group is:

$$246 \times 320 \times 59 \times 76 \times 112 \times 133 \times 17 \times 19 \times 23 \times 29 \times 31 \times 41 \times 47 \times 59 \times 71$$
$$= 808,017,424,794,512,875,886,459,904,961,710,757,005,754,368,000,000,000$$

The Monster group can be defined in terms of twelve symbols and eighty relations.

The relations are best described by use of a diagram (called a Coxeter diagram)

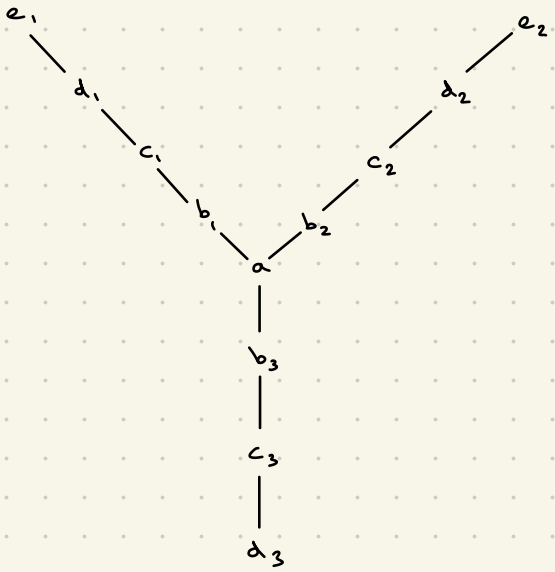


The relations are $(a)^2 = e$, $(b_i)^2 = e$, $(c_i)^2 = e$, $(d_i)^2 = e$, $(e_i)^2 = e$ (12 relations)

If connected on the diagram then their product cubed is the identity e.g. $(ab_i)^3 = e$, $(b_ic_i)^3 = e$ etc. (11 relations)

If not connected then they commute e.g. $ac_i = c_ia$, $ad_i = d_ia$ etc. (55 relations).

The spider relation $(ab_1c_1ab_2c_2ab_3c_3)^{10} = e$ (1 relation)



Monster relations:

Each symbol squares to the identity.

$a^2 = e$, $(b_i)^2 = e$, $(c_i)^2 = e$, $(d_i)^2 = e$, $(e_i)^2 = e$ [12 relations]

If connected on the diagram then their product cubed is the identity.

e.g. $(ab_i)^3 = e$, $(b_ic_i)^3 = e$ etc. [11 relations]

If not connected they commute.

e.g. $ac_i = c_ia$, $ad_i = d_ia$ etc. [55 relations]

The spider relation.

$(ab_1c_1ab_2c_2ab_3c_3)^{10} = e$. [1 relation]

Quotienting out the centre.

[1 relation].

Thank-you.