The Mesoamerican School on Modern Mathematics in Theoretical Physics: Lie algebras, geometry and their generalisations

An introduction to Lie algebras

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Lecture I: From (Lie) groups to algebras and back again.

Aims:

- The exponential map,
- the Adjoint and the adjoint actions and
- the Lie algebra.

Preliminary Comments.

Books:

- Lie Groups, Lie Algebras and Representations: An Elementary Introduction by Brian C. Hall. (page 129)

  (See also https://arxiv.org/abs/math-ph/0005032)

- Lie groups, Physics and Geometry by Robert Gilmore (page 148)

- Matrix Groups: An Introduction to Lie Group Theory by Andrew Baker.

  (See also http://www.maths.gla.ac.uk/~ajb/dvi-ps/lie-bern.pdf)

- An Introduction to Lie algebras and Representation Theory by James E. Humphreys.

  (page 35)

Preliminary Material: Matrix Groups.

A group is a set G equipped with a multiplication law mapping $G \times G \to G$ which we will denote by juxtaposition e.g. $g_1, g_2 \in G$ then $g_1 g_2 \in G$ (closure) and satisfying:

- $g_1(g_2 g_3) = (g_1 g_2) g_3 \quad \forall \; g_1, g_2, g_3 \in G$ (associativity)
- $\exists$ a unique element $e \in G$ such that $eg = g = ge$ called the identity and
- for each $g \in G$ $\exists$ a unique inverse element g^{-1} such that $gg^{-1} = e = g^{-1}g$

Most finite groups of low order are abelian which is motivation enough to investigate groups whose elements are matrices. Let $M_n(\mathbb{F})$ denote the square n-by-n matrices whose entries lie in $\mathbb{F}$ (typically $\mathbb{R}$ or $\mathbb{C}$). Then because a group must contain inverse elements we restrict attention to $GL(n,\mathbb{F}) = \{A \in M_n(\mathbb{F}) \mid \det A \neq 0\}$. We then seek subgroups of $GL(n,\mathbb{F})$ such as $SL(n,\mathbb{F}) = \{A \in GL(n,\mathbb{F}) \mid \det A = 1\}$. We are particularly drawn to sets of matrices defined by a relation on their transpose giving an inverse element

as $(g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$ and $(AB)^T = B^T A^T$

i.e. both transpose and inverse change the order of operation on a pair of group elements or matrices.

Hence we define $O(N) \equiv \{A \in GL(N,\mathbb{R}) \mid A^T = A^{-1}\}$ $\quad SO(N) \equiv \{A \in O(N) \mid \det A = 1\}$

$U(N) \equiv \{A \in GL(N,\mathbb{C}) \mid A^\dagger = A^{-1}\}$ $\quad SU(N) \equiv \{A \in U(N) \mid \det A = 1\}$

Now there are many other matrix groups, but these are the ones we will be most interested in.

Unlike a finite group, a matrix group has an infinite number of elements.

Instead of an order we have the dimension of the group, the number of real parameters needed to specify an element in the group e.g. $SO(2) \equiv \left\{R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \theta \in \mathbb{R}\right\}$ is one-dimensional.

As the parameter θ varies so does $R(\theta)$, we can speak of a continuous path in the matrix group, which opens matrix groups up to all the tools of analysis.

A matrix group is an example of a Lie group. There is a smooth manifold associated with the Lie group such that each element in the Lie group is given by a point in the manifold and pairs of points may be multiplied using the group multiplication law.

The Structures of SO(2), SO(3) and SU(2).

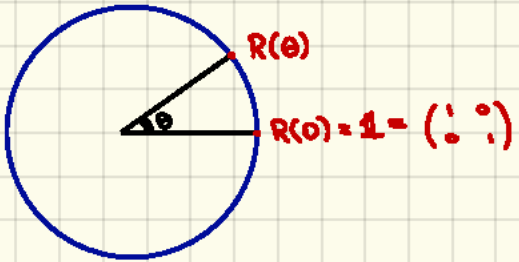
Each element of SO(2) is specified by $\theta \in \mathbb{R}$ with a periodic identification e.g. $\theta \sim \theta + n2\pi$ $n \in \mathbb{Z}$.

The points θ may be conveniently thought to lie on $S^1 \subset \mathbb{R}^2$ using the isomorphism

$$\phi \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = (\cos\theta, \sin\theta) \quad \text{for } \theta \in [0, 2\pi) \text{ with } \theta \sim \theta + n2\pi.$$

So we can see SO(2) as S^1

SO(2)



$SO(3)$ is the group of rotations in $\mathbb{R}^3$. Each rotation is specified by a vector v whose direction specifies the axis about which the rotation takes place and whose length $|v|$ encodes the rotation angle.



Now every rotation in $SO(3)$ is encoded by a vector/point in the ball of radius π $B_\pi(0) \in \mathbb{R}^3$, but this overcounts as points on the ball which are diametrically opposed give the same rotation.

So $SO(3)$ has the structure of $B_\pi(0)/\sim \in \mathbb{R}^3$ with $\sim$ identifying diametrically opposed points.

To construct $SU(2)$ as a manifold, first let $A \in M_{n}(\mathbb{C})$ satisfying both

$$A^{\dagger} = A^{-1} \quad \text{and} \quad \det(A) = 1.$$

Writing $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ we have

$$\begin{pmatrix} \alpha^{*} & \gamma^{*} \\ \beta^{*} & \delta^{*} \end{pmatrix} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \qquad \text{and} \qquad \alpha\delta - \beta\gamma = 1$$

So $-\beta = \gamma^{*}$ and $\alpha^{*} = \delta \Rightarrow \alpha\delta - \beta\gamma = |\alpha|^{2} + |\beta|^{2} = 1$

and writing $\alpha = x + iy$, $\beta = u + iv$ we have $x^{2} + y^{2} + u^{2} + v^{2} = 1$.

Which defines $S^{3} \in \mathbb{R}^{4}$ and each point on S^{3} reconstructs an $SU(2)$ matrix,

e.g. the point $x = 1$, $y = 0$, $u = 0$, $v = 0$ gives

$$\alpha = 1, \quad \beta = 0, \quad \delta = \alpha^{*} = 1, \quad \gamma = -\beta^{*} = 0$$

and $A = \begin{pmatrix} 1. & 0. \\ 0. & 1. \end{pmatrix} \in SU(2)$.

1.1 Infinitesimal Translation Generators.

Let $f(x) : \mathbb{R} \to \mathbb{R}$ be a continuous function and consider an infinitesimal space translation of its argument:

$$x \longrightarrow x + dx$$

We have,

$$\begin{aligned} f(x+dx) &= f(x) + \frac{df}{dx}\, dx + O(dx^2) \\ &= \left(1 + dx \frac{d}{dx}\right) f(x) + O(dx^2). \end{aligned}$$

Repeating the translation gives

$$\begin{aligned} f(x+2dx) &= \left(1 + dx \frac{d}{dx}\right) f(x+dx) + O(dx^2) \\ &= \left(1 + dx \frac{d}{dx}\right)^2 f(x) + O(dx^2). \end{aligned}$$

Repeating the translation n times and letting $n \to \infty$ such that $a \equiv n\,dx$ is a finite translation parameter gives

$$f(x+a) = \lim_{n\to\infty}\left(\left(1 + dx\frac{d}{dx}\right)^n f(x) + O(dx^2)\right)$$

$$= \lim_{n\to\infty}\left(\left(1 + \left(\frac{a}{n}\right)\frac{d}{dx}\right)^n f(x) + O\left(\left(\frac{a}{n}\right)^2\right)\right)$$

$$= \exp\left(a\frac{d}{dx}\right) f(x).$$

N.B. $\lim_{n\to\infty}\left(1 + \frac{a}{n}\frac{d}{dx}\right)^n = \exp\left(\lim_{n\to\infty}\left(n\ln\left(1 + \frac{a}{n}\frac{d}{dx}\right)\right)\right)$

Or try the binomial expansion.

$$= \exp\left(\lim_{m\to 0^+}\left(\frac{\ln\left(1 + am\frac{d}{dx}\right)}{m}\right)\right)$$

as $\lim_{h\to 0}\left(\frac{\ln(1+h)}{h}\right) = 1$

$$= \exp\left(a\frac{d}{dx}\right)$$

We have that $e^{\left(a\frac{d}{dx}\right)} f(x) = f(x+a)$.

Some Notation.

Let us define $X \equiv \frac{d}{dx}$ to be the "generator of translations".

It is the coefficient of the term in e^{aX} which is linear in the translation parameter a:

$$e^{(aX)} = 1 + aX + \frac{1}{2}(aX)^2 + \frac{1}{3!}(aX)^3 + \ldots$$

Element of the Lie algebra i.e. the linear part of Lie group element.

For many groups the Lie algebra is sufficient to reconstruct the group (more on this later).

A Bit More Notation.

Let us adopt $T_x(a) \equiv e^{(a\frac{d}{dx})}$ for a translation in x by a.

This notation is useful as we may readily adapt it to other translations:

$$T_y(b) \equiv e^{b\frac{d}{dy}} \equiv e^{bY} \qquad y \xrightarrow{T_y(b)} y+b$$

$$T_z(c) \equiv e^{c\frac{d}{dz}} \equiv e^{cZ} \qquad z \xrightarrow{T_z(c)} z+c$$

$$T_t(\tau) \equiv e^{\tau\frac{d}{dt}} \equiv e^{\tau W} \qquad t \xrightarrow{T_t(\tau)} t+\tau.$$

Note that each new translation has its own generator: Y, Z, W.

Example: Calculate the electrostatic potential of a point charge at $\underline{r}_q = \begin{pmatrix} 0 \\ 0 \\ d \end{pmatrix}$ which is denoted by $\phi(\underline{r} - \underline{r}_q)$ where $\phi(\underline{r}) = \frac{q}{|\underline{r}|}$.

$$\phi(\underline{r} - \underline{r}_q) = \exp(-\underline{r}_q \cdot \underline{\nabla})\,\phi(\underline{r}) \qquad \text{where } \underline{\nabla} \equiv \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$$

$$= \exp\left(-d\frac{\partial}{\partial z}\right)\phi(\underline{r})$$

$$= \left(1 - d\frac{\partial}{\partial z} + \frac{d^2}{2}\frac{\partial^2}{\partial z^2} - \dots\right)\phi(\underline{r})$$

Now, $\quad \frac{\partial \phi}{\partial z} = -\frac{q}{r^2}\frac{\partial z}{\partial r} = -\frac{qz}{r^3} = -\frac{q\cos\theta}{r^2}$ $\qquad z \equiv r\cos\theta$ (polar coords)

$$\frac{\partial^2 \phi}{\partial z^2} = \frac{\partial}{\partial z}\left(-\frac{qz}{r^3}\right) = -\frac{q}{r^3} + \frac{3z^2}{r^5} = -\frac{q}{r^3} + \frac{3\cos^2\theta}{r^3}.$$

The multipole expansion.

$$\phi(\underline{r} - \underline{r}_q) = \frac{q}{r}\left(1 + \frac{d\cos\theta}{r} + \frac{d^2(3\cos^2\theta - 1)}{2r^2} + \dots\right)$$

monopole term. dipole term. quadrupole term.

1.2. Exponentiation and the Group.

Let us confirm that the set

$$\left\{ \exp\left(a \frac{d}{dx}\right) \equiv T_x(a) \;\middle|\; a \in \mathbb{R} \right\}$$

does form a group acting on the space of continuous functions.

- $\exp\left(a_1 \frac{d}{dx}\right) \exp\left(a_2 \frac{d}{dx}\right) f(x) = f(x + a_1 + a_2) = \exp\left((a_1 + a_2) \frac{d}{dx}\right) f(x)$ Closure.

- the identity element is given when $a = 0$, i.e. $\exp\left(0 \frac{d}{dx}\right) f(x) = f(x+0) = f(x)$

- the inverse element to $\exp\left(a \frac{d}{dx}\right)$ is $\exp\left(-a \frac{d}{dx}\right)$

- the composition of exponential maps is associative.

This is the group of translations in one-dimension, it is isomorphic to $\mathbb{R}$.

Consider the example where $f(x) = x$ for $x \in \mathbb{R}$ then

$$T_x(a) : \mathbb{R} \to \mathbb{R} \quad \text{given by} \quad (T_x(a))(x) = x + a$$

i.e. $(T_x(a))(x)$ is a curve in $\mathbb{R}$ connected to the identity.

Note that each point in $\mathbb{R}$ can be associated with a translation by $(T_x(a))(0) = a$, so that each point in $\mathbb{R}$ corresponds to a translation of the identity, the identity element being zero. In this way we understand the geometrical structure of the translation group: it is $\mathbb{R}$.

The exponentiation of the Lie algebra generates a curve in the Lie group which is connected to the identity.

Translations in $\mathbb{R}^2$.

Consider the real algebra spanned by two generators X and Y.

It consists of all real linear sums of X and Y:

$$aX + bY \qquad \text{where} \quad a, b \in \mathbb{R}.$$

By exponentiation we obtain:

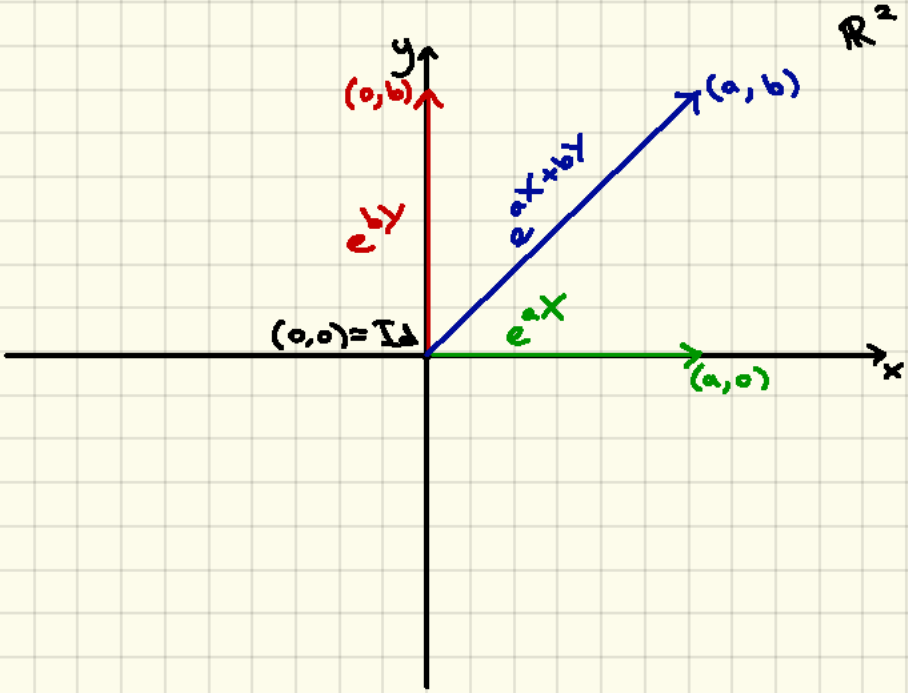
e^{aX} — a translation of a in the x-direction

e^{bY} — a translation of b in the y-direction.

Their action is defined on continuous functions of two variables and forms an abelian group as:

$$e^{aX}\, e^{bY}\, F(x,y) = F(x+a, y+b) = e^{bY}\, e^{aX}\, F(x,y).$$

As it is abelian we can confirm by expanding the exponential that $e^{aX}\, e^{bY} = e^{aX+bY}$.



The exponentiation of the Lie algebra covers the part of the group connected to the identity.

The Lie algebra spans the tangent space at the identity.

Definition: Let G be a Lie group and $\mathfrak{g}$ be a basis for the tangent space to the identity element of G.

The exponential map is a map, $\exp: \mathfrak{g} \to G$.

Remarks.

- We have considered the case where the algebra is a differential operator, X. For e^{aX} acting on smooth functions e^{aX} is well-defined.
- When X is a matrix $e^{aX} = \sum_{n=0}^{\infty} \frac{1}{n!}(aX)^n$ is a convergent series. N.B. X^n is computed by matrix multiplication.
- $e^0 = \mathbb{1}$, the identity element in G.
- If $XY = YX$ then $e^{X+Y} = e^X e^Y = e^Y e^X$.

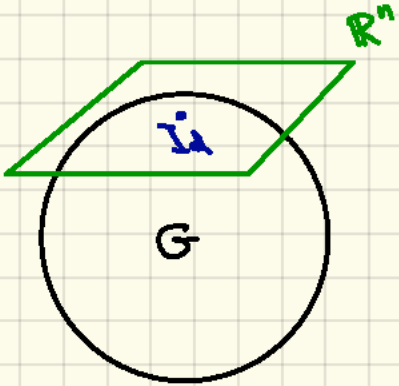
- $\frac{d}{d\theta}\left(e^{\theta X}\right)\Big|_{\theta=0} = X$. Given the grmp elements $e^{\theta X}$ with the transformation controlled by θ this gives a standard way to derive the corresponding algebra element X.

- Where X is a matrix, $\det(e^X) = \text{Trace}(X)$.

- $\gamma(\theta) \equiv e^{\theta X}$ is a smooth curve in G parameterised by θ which is the unique solution of the equation:

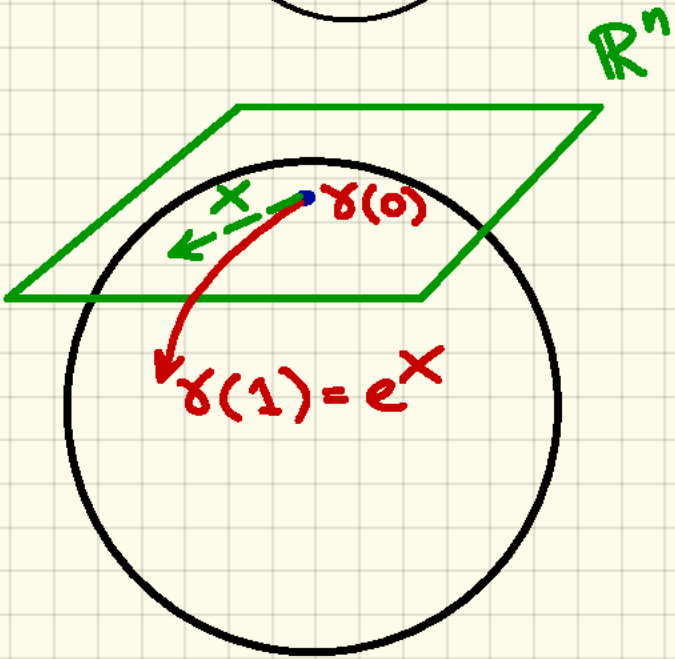
$$\frac{d\gamma}{d\theta} = X\gamma \qquad \text{where} \quad \gamma(0) = \mathbb{1}, \text{ the identity.}$$

The tangent space at the identity is the vector space $\mathbb{R}^n$ where $n = \dim(G)$.



$\gamma : [0,1] \to G$ given by $\gamma(\theta) = e^{\theta X}$ is the curve in G found by solving

$$\frac{d\gamma}{d\theta} = X\gamma$$



1.3 The Adjoint & adjoint Actions.

The Invariance of Physical Law.

Suppose that a system at position $\underline{r}$ and time t is described by a function $F(\underline{r},t)$ satisfying an equation of motion/state:

$$D F(\underline{r},t) = 0$$

where D is some differential operator.

Under a time translation we have:

$$\begin{aligned} 0 &= T_t(\tau)(DF(\underline{r},t)) \\ &= T_t(\tau)\left(D\, T_t(-\tau)T_t(\tau)\, F(\underline{r},t)\right) \\ &= \left(T_t(\tau)\, D\, T_t(-\tau)\right) F(\underline{r},t+\tau) \end{aligned}$$

Hence if $T_t(\tau)\, D(T_t(\tau))^{-1} = D$ then the form of the equation is invariant under time translations.

i.e. $[T_t(r), D] = 0$ as $T_t(r) D (T_t(r))^{-1} = D$

Similarly if $[e^{\underline{a}\cdot\underline{\nabla}}, D] = 0$ then the form of the equation is invariant under space translations.

We can reverse this argument to construct plausible equations of motion and find D such that $[D, \frac{\partial}{\partial t}] = 0 = [D, \underline{\nabla}]$.

E.g. $D = \underline{\nabla}$ then $[D, \underline{\nabla}] = 0$

$D = \underline{\nabla}^2$ then $Df = 0$ is invariant under $\underline{r} \to -\underline{r}$

$D = \underline{\nabla}^2 - \underline{k}^2$ where $\underline{k}$ is constant is more general still.

$D = W^2 - \omega^2$ where ω is constant is the analogous operator for time translations

Their combination (with dimensional analysis) gives the wave equation:

$$\left(\underline{\nabla}^2 - \frac{W^2}{c^2}\right) f(\underline{r}, t) = \left(\underline{k}^2 - \frac{\omega^2}{c^2}\right) f(\underline{r}, t).$$

Notice that for $D = \frac{\partial}{\partial t}$ then D is in the Lie algebra of the translation group and $T_t(\tau)\, D(T_t(\tau))^{-1}$ is a conjugate action of a group on its own Lie algebra to give back another element in the algebra. It is a general phenomenon.

Consider $Z \equiv e^{aX} Y e^{-aX}$ where $Y, X \in \mathfrak{g}$, the Lie algebra.

We will show that $Z \in \mathfrak{g}$ too as:

$$e^Z = 1 + Z + \tfrac{1}{2} Z^2 + \tfrac{1}{3!} Z^3 + \dots$$

Noting that

$$Z^n = (e^{aX} Y e^{-aX})^n = \overbrace{(e^{aX} Y \cancel{e^{-aX}})(\cancel{e^{aX}} Y \cancel{e^{-aX}}) \dots (\cancel{e^{aX}} Y e^{-aX})}^{n \text{ times}}$$

$$= e^{aX} Y^n e^{-aX}.$$

Then

$$e^Z = e^{aX}\left(1 + Y + \tfrac{1}{2} Y^2 + \tfrac{1}{3!} Y^3 + \dots\right) e^{-aX}$$

$$= e^{aX} e^{Y} e^{-aX} \in G \quad \text{hence } Z \in \mathfrak{g}.$$

Definition: Let G be a Lie group with Lie algebra $\mathfrak{g}$. The **Adjoint map** $Ad_g : \mathfrak{g} \to \mathfrak{g}$ is defined for each $g \in G$ by

$$Ad_g(X) = gXg^{-1}$$

where $X \in \mathfrak{g}$.

Remark

The Adjoint action of group on its own Lie algebra is not as surprising as it first appears. Recall that for any group G the inner automorphism is a map from G to itself defined by conjugation:

$$\phi_g : G \to G \text{ by } \phi_g(g') = gg'g^{-1}$$

Writing $g = e^{aX}$, $g' = e^{bY}$ we have $gg'g^{-1} = e^{aX}e^{bY}e^{-aX}$.

Now taking $\frac{d}{db}\Big|_{b=0}$ gives $e^{aX}Ye^{-aX} \in \mathfrak{g}$.

Additionally we understand $e^{aX} Y e^{-aX} = Z$ as a group transformation parameterised by a. What does the Adjoint action correspond to in the Lie algebra?

$$\frac{d}{da}\left(e^{aX} Y e^{-aX} \right)\Big|_{a=0} = \left(X e^{aX} Y e^{-aX} + e^{aX} Y (-X) e^{-aX} \right)\Big|_{a=0}$$

$$= XY - YX.$$

$$= [X, Y]$$

Definition: Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra.

The adjoint action, $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is given by

$$\mathrm{ad}_X(Y) = [X, Y].$$

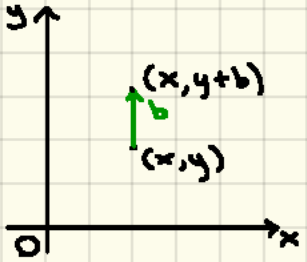
N.B. if G is a matrix Lie group of dimension n then

$$\mathrm{ad}_X : M_n(\mathbb{C}) \to M_n(\mathbb{C})$$

Example: The adjoint actions for $\mathbb{R}^2$ are trivial.

Group inner automorphism:

$$e^{aX} e^{bY} e^{-aX} (x,y) = e^{aX} e^{bY} (x-a, y)$$
$$= e^{aX} (x-a, y+b)$$
$$= (x, y+b)$$



N.B. $\mathbb{R}^2$ is abelian so $e^{aX} e^{bY} e^{-aX} = e^{aX} e^{-aX} e^{bY} = e^{bY}$.

Adjoint action (take $\frac{d}{db}\big|_{b=0}$)

$$e^{aX} Y e^{-aX} (x,y) = (0, 1)$$

i.e. $e^{aX} Y e^{-aX} = Y$ as $\mathbb{R}^2$ is abelian.

adjoint action (take $\frac{d}{da}\big|_{a=0}$ on last expression)

$$(XY - YX)(x,y) = (0,0) \quad \Rightarrow \quad [X,Y] = 0.$$

Example: The rotation group SO(3).

The rotation group in 3D is generated by

$$R_x(\theta) = \exp\left(\theta\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)\right) \equiv e^{\theta J_x}$$

$$R_y(\theta) = \exp\left(\theta\left(-z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}\right)\right) \equiv e^{\theta J_y} \quad \text{and}$$

$$R_z(\theta) = \exp\left(\theta\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\right) \equiv e^{\theta J_z}$$

N.B. each of the above are defined by their action on (x,y,z) e.g.

$$R_x(\theta)(x,y,z) = \exp(\theta J_x)(x,y,z)$$

$$= \left(\mathbb{1} + \theta J_x + \tfrac{1}{2}\theta^2 J_x^2 + \tfrac{1}{3!}\theta^3 J_x^3 + \dots\right)(x,y,z)$$

Now $J_x^2 = -z\partial_z - y\partial_y = -\mathrm{Id}_{(yz)}$, $J_x^3 = -J_x$, $J_x^4 = \mathrm{Id}_{(yz)}$ etc.

$$\therefore \quad R_x(\theta)(x,y,z) = \left(\mathbb{1} + \theta J_x - \tfrac{1}{2}\theta^2 \mathrm{Id}_{(yz)} - \tfrac{1}{3!}\theta^3 J_x + \tfrac{1}{4!}\theta^4 \mathrm{Id}_{(yz)} - \dots\right)(x,y,z)$$

$$= \left(x\partial_x + \mathrm{Id}_{yz}\cos\theta + J_x \sin\theta\right)(x,y,z)$$

$$= (x,\ y\cos\theta + z\sin\theta,\ z\cos\theta - y\sin\theta).$$

Now an example of the Adjoint action of $SO(3)$ on $so(3)$ is:

$$e^{\theta J_x} J_y e^{-\theta J_x} = e^{\theta J_x}\left(-z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}\right)\left(x\partial_x + (y\partial_y + z\partial_z)\cos\theta - \sin\theta\,(z\partial_y - y\partial_z)\right)$$

$$= e^{\theta J_x}\left(-z\partial_x + x\partial_z\cos\theta - x\partial_y\sin\theta\right)$$

$$= \left(x\partial_x + (y\partial_y + z\partial_z)\cos\theta + \sin\theta\,(z\partial_y - y\partial_z)\right)\left(-z\partial_x + x\partial_z\cos\theta - x\partial_y\sin\theta\right)$$

$$= (x\partial_z - z\partial_x)\cos\theta + (-x\partial_y + y\partial_x)\sin\theta.$$

$$\therefore \mathrm{Ad}_{J_x}(J_y) = J_y\cos\theta + J_z\sin\theta \in so(3)$$

adjoint action of J_x on J_y (take $\left.\frac{d}{d\theta}\right|_{\theta=0}$ on the expression above)

$$\mathrm{ad}_{J_x}(J_y) = J_z$$

and $[J_x, J_y] = (z\partial_y - y\partial_z)(x\partial_z - z\partial_x) - (x\partial_z - z\partial_x)(z\partial_y - y\partial_z)$

$$= y\partial_x - x\partial_y$$

$$= J_z.$$

Definition: A **Lie algebra** is a vector space V together with a bilinear map $[\,]: V \times V \to V$ called the Lie bracket such that for all $u, v, w \in V$:

(i.) $[uv] = -[vu]$ and

(ii.) $[u[vw]] + [v[wu]] + [w[uv]] = 0$.

Comments.

- For matrix Lie groups the Lie bracket is the matrix commutator i.e. $[AB] \equiv [A,B] = AB - BA$
- The statement in part (ii.) is called the Jacobi identity and is satisfied for $[AB] = [A,B]$ automatically as:

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]] = [A, BC-CB]+[B, CA-AC]+[C, AB-BA]$$

$$= A(BC-CB) - (BC-CB)A$$

$$+B(CA-AC) - (CA-AC)B$$

$$+C(AB-BA) - (AB-BA)C .$$

$$= 0.$$

We will specialise to matrix groups herein and use the matrix Lie bracket.

We will make the connection with the operator algebras we constructed later on.

Summary

- A Lie group is a differentiable manifold equipped with a group multiplication rule.
- A Lie algebra is a linearisation of the group and spans the tangent space ($\mathbb{R}^n$ where $n = \dim(G)$) at the identity.

  e.g. $e^{aX} \in G$ for $X \in \mathfrak{g}$, $a \in \mathbb{R}$

  and hence $X = \frac{d}{da}\left(e^{aX}\right)\Big|_{a=0}$.
- A Lie group has a natural action (the Adjoint action) on the vector space spanned by its own Lie algebra: $Ad_g(X) = gXg^{-1} \in \mathfrak{g}$
- The corresponding Lie algebra action is the adjoint action

$$ad_X(Y) = [X, Y] \in \mathfrak{g}.$$

Lecture II: The Semisimple Lie Algebras.

Aims:

- The algebras of some matrix Lie groups,

  with occasional examples of relevance to physics.

The Lie algebras of $GL(N,\mathbb{C})$ and $GL(N,\mathbb{R})$.

Let $e^{\theta X} \in GL(N,\mathbb{C})$ then $\det\left(e^{\theta X}\right) \neq 0$ which does not give a restriction on $X \in M_N(\mathbb{C})$.

Near the identity element we have

$$\exp(\theta X) = \mathbb{1} + \theta X + O(\theta^2)$$

which for $\theta \ll 1$ approaches the map:

$$\gamma(\theta) = \mathbb{1} + \theta X.$$

Our aim is to find all X that solve $\det(\gamma(\theta)) \neq 0$ for small θ.

The roots satisfy $\det(\gamma(\theta)) = 0$

For $\theta \neq 0$ this is equivalent to solving

$$\det(\theta^{-1}\mathbb{1} + X) = 0$$

Which is satisfied when $\theta = -1/\lambda$ where λ is an eigenvalue of X.

Hence for $|\theta| < \frac{1}{|\lambda|}$ the equation $\det(\gamma(\theta)) \neq 0$ is satisfied for all $X \in M_n(\mathbb{C})$.

Similar arguments hold to show that $gl(N,\mathbb{R}) \cong M_n(\mathbb{R})$

$$\dim(gl(N,\mathbb{C})) = 2N^2$$

$$\dim(gl(N,\mathbb{R})) = N^2.$$

N.B. singular matrices $X \in M_n(\mathbb{C})$ exponentiate to elements of $GL(N,\mathbb{C})$ as if $\det(X) = 0$ then $\det(e^{\theta X}) = \det(\mathbb{1} + \theta X + \frac{\theta^2}{2!}X^2 + \dots)$

$$= 1 + \theta^0 \det X + O((\det X)^2) = 1.$$

The Lie algebras of $SL(N,\mathbb{C})$ and $SL(N,\mathbb{R})$.

Let $e^{\theta X} \in SL(N,\mathbb{C})$

$$\Rightarrow \det\left(e^{\theta X}\right) = \exp\left(\mathrm{Tr}(\theta X)\right) = 1$$

$\Rightarrow X$ are traceless matrices with $X \in M_N(\mathbb{C})$

$\dim\left(sl(N,\mathbb{C})\right) = 2N^2 - 2$

Similarly, $sl(N,\mathbb{R}) = \left\{ X \in M_N(\mathbb{R}) \mid \mathrm{Tr}\, X = 0 \right\}$

$\dim\left(sl(N,\mathbb{C})\right) = N^2 - 1.$

$\det(e^X) = e^{\mathrm{Tr}\, X}$ – if X is diagonalizable then $X = T\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} T^{-1}$

then $e^X = T\begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix} T^{-1}$

hence $\mathrm{Tr}(X) = \sum \lambda_i$ and $\det(e^X) = e^{\sum \lambda_i} = e^{\mathrm{Tr} X}$

– if X is nilpotent then $X = T\begin{pmatrix} 0 & * & \\ & \ddots & * \\ & & 0 \end{pmatrix} T^{-1}$ so $\mathrm{Tr}\, X = 0$

while $e^X = T\begin{pmatrix} 1 & * & \\ & \ddots & * \\ & & 1 \end{pmatrix} T^{-1} \Rightarrow \det(e^X) = 1$ and $e^{\mathrm{Tr} X} = 1$.

– $X = S + N$, S is diagonalizable, N is nilpotent & $SN = NS$

$\therefore \det(e^X) = \det(e^S e^N) = \det(e^S)\det(e^N) = e^{\mathrm{Tr} S} = e^{\mathrm{Tr} X}$.

Example: $sl(2,\mathbb{R})$

$$sl(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a,b,c \in \mathbb{R} \right\}$$

The algebra is three-dimensional with a basis

$$H \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \quad F \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

We can define the algebra by stating all of its nontrivial commutators

$$[H,E] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2E$$

$$[H,F] = -2F$$

$$[E,F] = H.$$

More abstractly given a basis X_i for a Lie algebra we can define

$$[X_i, X_j] = C_{ij}^{k} X_k$$

where the repeated index is summed over. The $C_{ij}^{k} = -C_{ji}^{k}$ are called the structure constants of the algebra.

The Lie Algebras of $O(N)$ and $SO(N)$.

Let $e^{\theta X} \in O(N)$ then

$$(e^{\theta X})(e^{\theta X})^T = \mathbb{1}$$

i.e. $\left(\mathbb{1} + \theta X + \frac{\theta^2}{2} X^2 + \dots\right)\left(\mathbb{1} + \theta X + \frac{\theta^2}{2!} X^2 + \dots\right)^T = \mathbb{1}$

$$\therefore \mathbb{1} + \theta (X + X^T) + O(\theta^2) = \mathbb{1}$$

$$\Rightarrow X + X^T = 0$$

$$\text{or} \quad X^T = -X$$

That is $o(N)$ consists of the skew-symmetric matrices in $M_N(\mathbb{R})$.

Hence $\dim(o(N)) = \frac{N}{2}(N-1)$.

$so(N) \subset o(N)$ consisting of the traceless, skew-symmetric matrices in $M_N(\mathbb{R})$

$\rightarrow$ the algebra of $SO(N)$ is the same as that of $O(N)$!

This is because $O(N)$ is disconnected and exponentiation generates only the rotations

Example: $so(2)$.

The traceless skew-symmetric matrices in $M_2(\mathbb{R})$ are

$$\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \qquad a \in \mathbb{R}.$$

The algebra is one-dimensional, having a basis element

$$X \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Noting that $X^2 = -\mathbb{1}$, $X^3 = -X$, $X^4 = \mathbb{1}$, ...

one may confirm that

$$\begin{aligned} \exp(\theta X) &= \mathbb{1} + \theta X + \frac{\theta^2}{2!} X^2 + \frac{\theta^3}{3!} X^3 + \frac{\theta^4}{4!} X^4 + \dots \\ &= \mathbb{1} + \theta X - \frac{\theta^2}{2!} \mathbb{1} - \frac{\theta^3}{3!} X + \frac{\theta^4}{4!} \mathbb{1} + \dots \\ &= \mathbb{1} \cos\theta + X \sin\theta \\ &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \\ &\in SO(2). \end{aligned}$$

Example: $so(3)$.

$$R_x(\theta) = \begin{pmatrix} 1. & 0. & 0. \\ 0. & \cos\theta. & -\sin\theta. \\ 0. & \sin\theta. & \cos\theta. \end{pmatrix}, R_y(\theta) = \begin{pmatrix} \cos\theta. & 0. & \sin\theta. \\ 0. & 1. & 0. \\ -\sin\theta. & 0. & \cos\theta. \end{pmatrix}, R_z(\theta) = \begin{pmatrix} \cos\theta. & -\sin\theta. & 0. \\ \sin\theta. & \cos\theta. & 0. \\ 0. & 0. & 1. \end{pmatrix}$$

generate $SO(3)$.

Evaluating $\frac{d}{d\theta}\Big|_{\theta=0}$ gives the algebra elements:

$$J_x = \begin{pmatrix} 0. & 0. & 0. \\ 0. & 0. & -1. \\ 0. & 1. & 0. \end{pmatrix}, J_y = \begin{pmatrix} 0. & 0. & 1. \\ 0. & 0. & 0. \\ -1. & 0. & 0. \end{pmatrix}, J_z = \begin{pmatrix} 0. & -1. & 0. \\ 1. & 0. & 0. \\ 0. & 0. & 0. \end{pmatrix}.$$

Which form a basis for the skew-symmetric matrices in $M_3(\mathbb{R})$.

$$\left.\begin{aligned} [J_x, J_y] &= J_z \\ [J_x, J_z] &= -J_y \\ [J_y, J_z] &= J_x \end{aligned}\right\} \quad [J_i, J_j] = \varepsilon_{ijk} J_k$$

by the abuse of notation: $x=1, y=2, z=3$.

Digression: Operator Algebras.

We already met $so(3)$ as an operator algebra – how are the matrix and operator formulations related?

Given a matrix expression A for a Lie algebra element one can form the operator $x_i A^i{}_j \partial^j$

then $[x_i A^i{}_j \partial^j, x_k B^k{}_l \partial^l] = x_i A^i{}_j \partial^j (x_k B^k{}_l \partial^l) - x_k B^k{}_l \partial^l (x_i A^i{}_j \partial^j)$

$= x_i A^i{}_j \delta^j_k B^k{}_l \partial^l + x_i A^i{}_j x_k B^k{}_l \partial^j \partial^l$

$- x_k B^k{}_l \delta^l_i A^i{}_j \partial^j - x_k B^k{}_l x_i A^i{}_j \partial^l \partial^j$

$= x_i A^i{}_k B^k{}_l \partial^l - x_k B^k{}_i A^i{}_j \partial^j$

$= x_i [A, B]^i{}_j \partial^j$

$= x_i C^i{}_j \partial^j$ where $[A, B] = C$.

The operator algebra $x_i A^i{}_j \partial^j$ satisfies the same Lie brackets as A.

e.g. for $so(3)$ we have

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

hence an operator algebra is

$$\hat{J}_x = (x, y, z) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = (x, y, z) \begin{pmatrix} 0 \\ -\partial_z \\ \partial_y \end{pmatrix} = -y\partial_z + z\partial_y.$$

$$\hat{J}_y = (x, y, z) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = (x, y, z) \begin{pmatrix} \partial_z \\ 0 \\ -\partial_x \end{pmatrix} = x\partial_z - z\partial_x.$$

$$\hat{J}_z = (x, y, z) \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = (x, y, z) \begin{pmatrix} -\partial_y \\ \partial_x \\ 0 \end{pmatrix} = -x\partial_y + y\partial_x.$$

Which reconstructs the differential operators we used previously as a basis for $so(3)$.

Health warning: other operator algebras are available!

The Lie Algebras of $U(N)$ and $SU(N)$.

Let $e^{\theta X} \in U(N)$

$$\Rightarrow \mathbb{1} = (e^{\theta X})(e^{\theta X})^\dagger$$

$$= (\mathbb{1} + \theta X + O(\theta^2))(\mathbb{1} + \theta X^\dagger + O(\theta^2))$$

$$= \mathbb{1} + \theta(X + X^\dagger) + O(\theta^2)$$

$$\Rightarrow X^\dagger = -X.$$

i.e. $u(N)$ is spanned by the skew-hermitian matrices in $M_N(\mathbb{C})$.

$$\therefore \dim(u(N)) = N + N(N-1) = N^2.$$

N imaginary entries on the diagonal.

$\frac{1}{2}N(N-1)$ complex numbers in the upper triangle (i.e. $2(\frac{1}{2}N(N-1))$ real numbers)





The Lie algebra of $SU(N)$ is the traceless part of $u(N)$,

$$su(N) = \{ X \in M_n(\mathbb{C}) \mid X^\dagger = -X, \operatorname{Tr}(X) = 0 \}$$

$\Rightarrow \dim(su(N)) = N^2 - 1.$

Example: $su(2)$.

Let $X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and now apply the defining relations.

$X^\dagger = -X \Rightarrow \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix} = -\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$

$\operatorname{Tr} X = 0 \Rightarrow \alpha + \delta = 0$

i.e. we have that $\alpha = -\delta$, $\operatorname{Re}(\alpha) = 0 = \operatorname{Re}(\delta)$

so let us write $\alpha = iz = -\delta$, $\beta = y + ix$ for $x, y, z \in \mathbb{R}$

then as $\gamma^* = -\beta$ we have $\gamma = (-\beta)^* = -y + ix$

$$\therefore X = \begin{pmatrix} iz & y+ix \\ -y+ix & -iz \end{pmatrix}$$

$$X = \begin{pmatrix} iz & y+ix \\ -y+ix & -iz \end{pmatrix}$$

$$= ix \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + iy \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + iz \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= i\underline{x} \cdot \underline{\sigma}$$

where $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\underline{\sigma} = \begin{pmatrix} \underline{\sigma}_x \\ \underline{\sigma}_y \\ \underline{\sigma}_z \end{pmatrix}$ and

$$\underline{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \underline{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \underline{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are called the Pauli matrices.

They satisfy $\sigma_i^2 = \mathbf{1}$, $\sigma_i \sigma_j = -\sigma_j \sigma_i = i\varepsilon_{ijk}\sigma_k$

Then $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$.

(We have abused our labelling notation again (!) $x \to 1$, $y \to 2$, $z \to 3$)

Example: $su(2) \cong so(3)$

From earlier computations we have that

$su(2)$ $\quad [\sigma_i, \sigma_j] = 2i\,\epsilon_{ijk}\,\sigma_k = C_{ijk}\,\sigma_k$

$so(3)$ $\quad [J_i, J_j] = \epsilon_{ijk}\,J_k = D_{ijk}\,J_k$

N.B. structure constants $C_{ijk} = 2i\,D_{ijk}$

Consider the map $\phi(J_i) = \frac{-i\sigma_i}{2}$

It is a homomorphism as

$$\phi[J_i, J_j] = \phi(\epsilon_{ijk}J_k) = \tfrac{-i}{2}\,\epsilon_{ijk}\,\sigma_k = -\tfrac{1}{4}[\sigma_i, \sigma_j] = [\phi(J_i), \phi(J_j)]$$

and it's a bijection, so it is an isomorphism of Lie algebras.

But although the Lie algebras are isomorphic, the groups are not, instead $\frac{SU(2)}{\mathbb{Z}_2} \cong SO(3)$.

One can use this Lie algebra isomorphism to construct a Lie group isomorphism:

$$\Phi\left(e^{\theta_i J_i}\right) \equiv e^{\theta_i \phi(J_i)} = e^{\theta_i\left(-i\frac{\sigma_i}{2}\right)} \in SU(2).$$

Notice that to cover $SU(2)$ $\theta_i \in [0, 4\pi)$

while the same range of θ_i corresponds to two revolutions about the x_i axis in $SO(3)$.

When $\theta_i = 2\pi + \theta_i'$ then $e^{\theta_i\left(-i\frac{\sigma_i}{2}\right)} = e^{-i\pi} e^{\theta_i'\left(-i\frac{\sigma_i}{2}\right)} = -\mathbb{1}\, e^{\theta_i'\left(-i\frac{\sigma_i}{2}\right)}$,

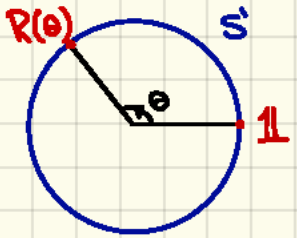
i.e. $-\mathbb{1} \in SU(2)$.

The same Lie algebra may exponentiate to different Lie groups. The range of the parameters required to return to the identity in each group differs.

The Universal Covering Group.

Recall that the exponential map is onto the part of a group connected to the identity.

E.g. $SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \ \theta \in \mathbb{R} \right\}$



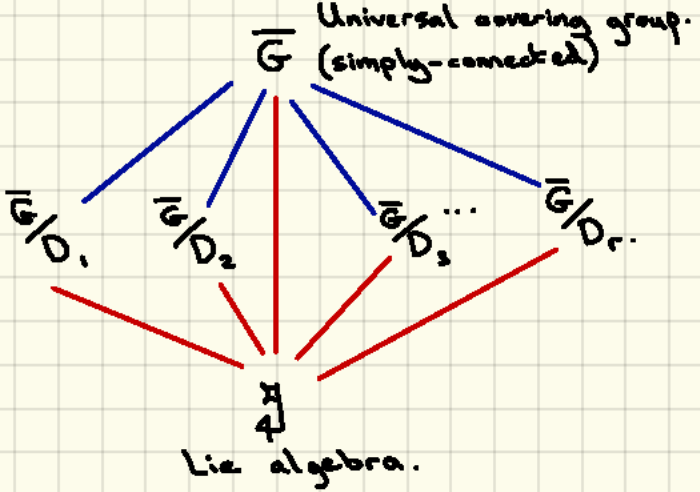
$SO(2)$ is connected: every path between two points lies in $SO(2)$

$SO(2)$ is not simply-connected: every closed path (loop) in $SO(2)$ cannot be smoothly contracted to the identity.

While there are many Lie groups associated to each lie algebra there is only one which is simply-connected.

For every Lie algebra there is a unique simply-connected group which is called the **universal covering group**, which we will denote $\bar{G}$.

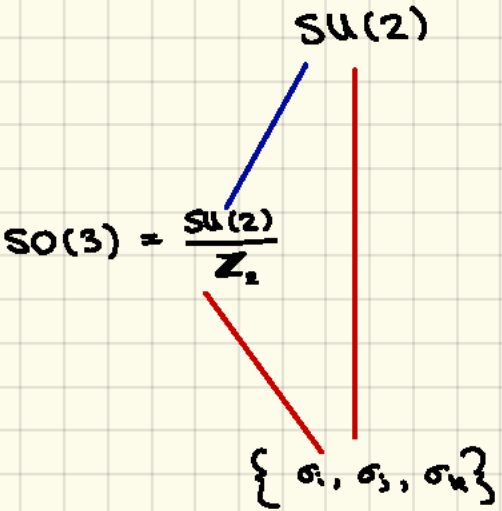
For example $SU(2)$ is simply-connected and is the universal covering group of $SO(3)$.



Cartan's covering theorem:
Lie groups sharing the same Lie algebra are all quotients of the universal covering group $\bar{G}$ by the discrete invariant subgroups of the centre of $\bar{G}$, denoted D_i.

N.B. $\bar{G}/D_i$ all are locally isomorphic to $\bar{G}$: they all have the same Lie algebra.

For example, when $\bar{G} = SU(2)$ the maximal H, consists of $\{1, -1\}$.

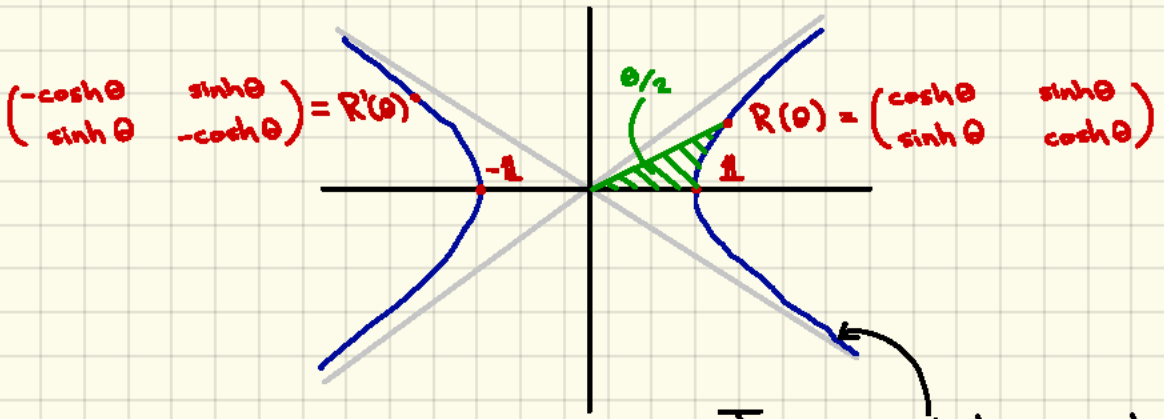


Some groups are disconnected, such as

$$SO(1,1) = \left\{ \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}, \begin{pmatrix} -\cosh\theta & \sinh\theta \\ \sinh\theta & -\cosh\theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\}$$

which is the group which preserves the Minkowski metric on $\mathbb{R}^2$ e.g.

points on $x^2 - y^2 = 1$ (For example are translated along the hyperbola)



The algebra is spanned by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

The exponential map only produces this branch of the group.

Summary.

- $\dim(gl(N,\mathbb{C})) = 2N^2$
  $\dim(sl(N,\mathbb{C})) = 2N^2 - 2$

  $\dim(gl(N,\mathbb{R})) = N^2.$
  $\dim(sl(N,\mathbb{R})) = N^2 - 1.$

  $\dim(o(N)) = \dim(so(N)) = \frac{N}{2}(N-1).$ $\qquad o(N) = so(N)$

  $\dim(u(N)) = N^2.$
  $\dim(su(N)) = N^2 - 1.$

- $sl(2,\mathbb{R})$ is spanned by $H \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $F \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

  $[H,E] = 2E$, $[H,F] = -2F$, $[E,F] = H.$

- $so(3)$: $[J_i, J_j] = \varepsilon_{ijk} J_k$

- $[\sigma_i, \sigma_j] = 2i\,\varepsilon_{ijk}\sigma_k.$

- $su(2) \cong so(3).$

Lecture III: Some Tools for Structure Theory.

- Some representation theory,
- The Cartan-Killing Form and
- Solvable, Nilpotent, Simple and Semisimple.

Representations of the Lie Algebra.

The Lie algebra is a basis for a vector space $\mathbb{R}^n$ in the vicinity of the group identity element.

Given $\mathbb{R}^n$ we may:

- act on it with linear transformations and
- construct an inner product.

We already know that the adjoint action gives a linear transformation of the Lie algebra:

$$ad_{\mathfrak{g}}: \mathfrak{g} \to End(\mathfrak{g}) \quad \text{given by} \quad ad_X(Y) = [X, Y] \quad \forall X, Y \in \mathfrak{g}$$

We can rewrite the adjoint action as a matrix action on a vector space whose basis vectors are the Lie algebra elements: giving a matrix $\pi(X)$ for each $X \in \mathfrak{g}$. These matrices are called the adjoint representation of $\mathfrak{g}$.

First we need some definitions. We will start with group representations.

Definition: Let G be a matrix Lie group. A **finite-dimensional real representation** of G is a Lie group homomorphism

$$\Pi : G \to GL(n, \mathbb{R}). \qquad \Pi(g_1 g_2) = \Pi(g_1)\Pi(g_2)$$

Definition: Let $\mathfrak{g}$ be a Lie algebra. A **finite-dimensional real representation** of $\mathfrak{g}$ is a Lie algebra homomorphism

$$\pi : \mathfrak{g} \to GL(n, \mathbb{R}). \qquad \pi([X, Y]) = [\pi(X), \pi(Y)]$$

Definition: If $\Pi(G)$ or $\pi(\mathfrak{g})$ are one-to-one homomorphisms, then the representation is called **Faithful**.

i.e. a representation is a way to recast a group or Lie algebra as a matrix action on a vector space.

Remarks.

- We could also consider complex representations

$$\pi : G \to GL(n, \mathbb{C})$$

or as an action on an abstract vector space V:

$$\pi : G \to GL(V).$$

- For each representation of G acting on V there is associated a unique Lie algebra representation on V:

$$\Pi(e^X) = e^{\pi(X)}$$

where $\pi(X) = \frac{d}{d\theta}\left(e^{\theta X}\right)\Big|_{\theta=0}$.

Note that $\Pi(e^{\theta X} e^{\phi Y} e^{-\theta X}) = \Pi(e^{\theta X})\Pi(e^{\phi Y})\Pi(e^{-\theta X}) \in G$ implies:

$\left(\frac{d}{d\phi}\Big|_{\phi=0}\right)$ $\Pi(e^{\theta X})\,\pi(Y)\,\Pi(e^{-\theta X}) \in \mathfrak{H}$ and

$\left(\frac{d}{d\theta}\Big|_{\theta=0}\right)$ $[\pi(X), \pi(Y)] = \pi[X, Y] \in \mathfrak{H}$.

Definition: Let π be a representation of G on a vector space V.

A subspace W of V is called **invariant** if

$$\pi(A)\,w \in W \qquad \forall\, A \in G.$$

A representation with no non-trivial ($W \neq \{0\}$ nor $W \neq V$) invariant subspaces is called **irreducible**.

Invariant and irreducible are defined similarly for representations of the Lie algebra.

Example.

Consider the representation of $SO(2)$ given by:

$$\pi(R(\theta)) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The z-direction (the space spanned by $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$) is an invariant subspace, hence this representation is not irreducible (it is reducible). It is faithful if $\theta \in [0, 2\pi)$.

We are interested in matrix Lie groups and algebras, which are already expressed as matrices - why do we wish to consider rewriting them as other matrices. The short answer is that it is interesting to know when we can do this, the longer answer is that it will prove useful.

We will focus on the Lie algebra where the first representation of interest is the adjoint representation (sometimes called the regular representation), given when the vector space is the Lie algebra:

$$[Z, X_i] = \pi(Z)_i^{\ j} X_j$$

where Z is an arbitrary element of $\mathfrak{g}$ and X_i is the basis of the Lie algebra.

Example: the adjoint representation of $\mathfrak{sl}(2,\mathbb{R})$.

Recall that $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ so that

$$[H, E] = 2E$$

$$[H, F] = -2F$$

$$[E, F] = H$$

Then ordering the algebra basis H, E, F we construct:

$$\pi(H) = \begin{pmatrix} 0. & 0. & 0. \\ 0. & 2. & 0. \\ 0. & 0. & -2. \end{pmatrix}$$

$$\pi(E) = \begin{pmatrix} 0. & 0. & 1. \\ -2. & 0. & 0. \\ 0. & 0. & 0. \end{pmatrix}$$

$$\pi(F) = \begin{pmatrix} 0. & -1. & 0. \\ 0. & 0. & 0. \\ 1. & 0. & 0. \end{pmatrix}$$

i.e. $[H, X_i] = \pi(H)_i^j X_j$

where $X_1 = H$, $X_2 = E$, $X_3 = F$.

$$\pi(aH + bE + cF) = \begin{pmatrix} 0. & -c. & b. \\ -2b. & 2a. & 0. \\ 2c & 0. & -2a. \end{pmatrix}$$

Example: $\pi(H_1)$ in the adjoint representation of $sl(3,\mathbb{R})$

As we will see the $sl(3,\mathbb{R})$ algebra satisfies:

$$[H_1, E_1] = 2E_1 \quad [H_1, E_2] = -E_2 \quad [H_1, E_3] = E_3.$$

$$[H_1, F_1] = -2F_2 \quad [H_1, F_2] = F_2 \quad [H_2, E_3] = E_3.$$

Let the basis be

$$X_1 = H_1,\ X_2 = E_1,\ X_3 = F_1,\ X_4 = H_2,\ X_5 = E_2,\ X_6 = F_2,\ X_7 = E_3,\ X_8 = F_3$$

$$\pi(H_1) = \begin{pmatrix} 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 2. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & -2. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & -1. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1. \end{pmatrix}$$

The adjoint representation of $sl(3)$ acts on an 8-dimensional vector space.

Example: the adjoint representation of $so(3)$.

$$[R_i, R_j] = \varepsilon_{ijk} R_k$$

$$\therefore \pi(aR_1 + bR_2 + cR_3) = \begin{pmatrix} 0. & -c. & b. \\ c. & 0. & -a. \\ -b. & a. & 0. \end{pmatrix}.$$

Complexification of Representations.

Proposition: Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Then every finite-dimensional representation π of $\mathfrak{g}$ has a unique extension to a complex linear representation of $\mathfrak{g}_{\mathbb{C}}$ also denoted π and given by

$$\pi(X+iY) = \pi(X) + i\pi(Y)$$

for all $X, Y \in \mathfrak{g}$.

N.B. if π is an irreducible representation of $\mathfrak{g}$ then it extends to an irreducible representation of $\mathfrak{g}_{\mathbb{C}}$.

Complexification results in the unification of some of the most fundamental Lie algebras.

$sl(2,\mathbb{R})$, $su(2)$ and $sl(2,\mathbb{C})$.

$sl(2,\mathbb{R})$ is spanned by $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ $\quad a,b,c \in \mathbb{R}$

$sl(2,\mathbb{C})$ is spanned by $\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ $\quad \alpha,\beta,\gamma \in \mathbb{C}$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} = \begin{pmatrix} a+ia' & b+ib' \\ c+ic' & -a-ia' \end{pmatrix} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} + i \begin{pmatrix} a' & b' \\ c' & -a' \end{pmatrix}$$

Hence $sl(2,\mathbb{R})_{\mathbb{C}} \cong sl(2,\mathbb{C})$.

$su(2)$ is spanned by $\begin{pmatrix} iz. & y+ix. \\ -y+ix. & -iz. \end{pmatrix}$ where $x, y, z \in \mathbb{R}$.

Now,

$$\begin{pmatrix} iz. & y+ix. \\ -y+ix. & -iz. \end{pmatrix} + i \begin{pmatrix} iz'. & y'+ix'. \\ -y'+ix'. & -iz'. \end{pmatrix} = \begin{pmatrix} -z'+iz. & (y-x')+i(x+y'). \\ -(y+x')+i(x-y') & z'-iz \end{pmatrix}$$

$$= \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \in sl(2,\mathbb{C})$$

Hence $su(2)_{\mathbb{C}} \cong sl(2,\mathbb{C})$.

The basis for $sl(2,\mathbb{C})$ is $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

but note that algebra is complex-linear.

Health warning: due to $sl(2,\mathbb{R})$ and $su(2)$ both complexifying to $sl(2,\mathbb{C})$ it is not uncommon for $su(2)$ and $sl(2)$ to be used interchangeably.

With this understanding that $sl(2,\mathbb{C})$ will be a central algebra, let us now construct an infinite set of its irreducible representations.

Representations of $sl(2,\mathbb{C})$

For this example we will return to operator algebras and denote

$$\pi_m(H) \equiv (z_1, z_2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \partial_{z_1} \\ \partial_{z_2} \end{pmatrix} = z_1\partial_{z_1} - z_2\partial_{z_2}$$

$$\pi_m(E) \equiv (z_1, z_2)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \partial_{z_1} \\ \partial_{z_2} \end{pmatrix} = z_1\partial_{z_2}$$

$$\pi_m(F) \equiv (z_1, z_2)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \partial_{z_1} \\ \partial_{z_2} \end{pmatrix} = z_2\partial_{z_1}$$

(The meaning of $m \in \mathbb{Z}$ will be clear soon).

This algebra acts on the space of smooth functions $f(z_1, z_2)$ of the complex variables z_1 and z_2. Consider the case where

$$f(z_1, z_2) = a_0 z_1^m + a_1 z_1^{m-1} z_2 + a_2 z_1^{m-2} z_2^2 + \dots + a_m z_2^m \qquad \text{for } m \in \mathbb{Z}^+.$$

The space V_m of such functions is $(m+1)$-dimensional as its basis is given by

$$z_1^{m-k} z_2^k \quad \text{for } k = 0, 1, 2, \dots, m.$$

The action of the algebra on the basis is:

$$\pi_m(H)\, z_1^{m-k} z_2^k = (z_1\partial_{z_1} - z_2\partial_{z_2})\, z_1^{m-k} z_2^k$$

$$= (m-k)\, z_1^{m-k} z_2^k - k\, z_1^{m-k} z_2^k$$

$$= (m-2k)\, z_1^{m-k} z_2^k.$$

i.e. $z_1^{m-k} z_2^k$ is an eigenvector for $\pi_m(H)$ with eigenvalue $(m-2k)$.

$$\pi_m(E)\, z_1^{m-k} z_2^k = (z_1\partial_{z_2})\, z_1^{m-k} z_2^k$$

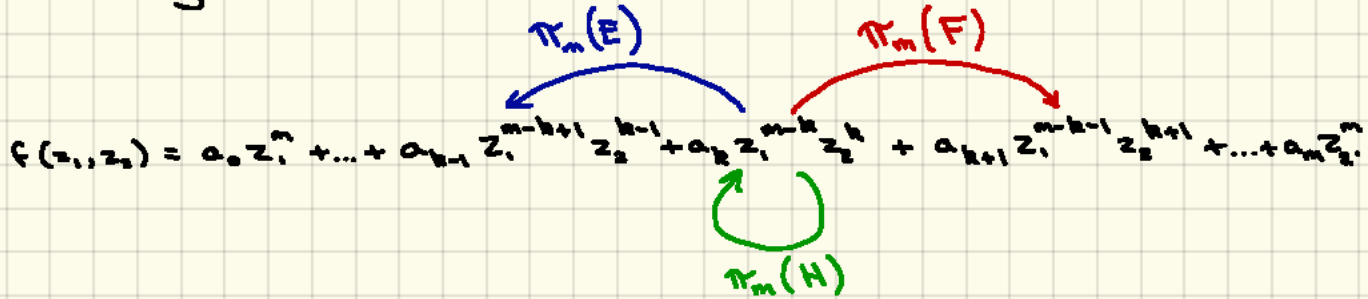
$$= k\, z_1^{m-k+1} z_2^{k-1}$$

$$\pi_m(F)\, z_1^{m-k} z_2^k = (z_2\partial_{z_1})\, z_1^{m-k} z_2^k$$

$$= (m-k)\, z_1^{m-k-1} z_2^{k+1}.$$

Hence $\pi_m(E)$ and $\pi_m(F)$ both map to basis vectors in this space.

Schematically:



$\pi_m(E)$ raises the power of z_1, $\pi_m(F)$ lowers the power of z_1.

i.e. $\pi_m(H)$, $\pi_m(E)$ and $\pi_m(F)$ can be written as $(m+1)$-by-$(m+1)$ matrices acting on V_m.

N.B.I. We have a representation of $sl(2,\mathbb{C})$ for every integer $m \in \mathbb{Z}^+$.

N.B.II. $\pi_m(E)(z_1^m) = z_1 \partial_{z_2}(z_1^m) = 0$ and $\pi_m(F)(z_2^m) = z_2 \partial_{z_1}(z_2^m) = 0$.

Each representation on V_m is irreducible.

Proposition: π_m is an irreducible representation of $sl(2,\mathbb{C})$.

Proof: We must show that every invariant subspace of π_m is trivial.

Suppose that $W \subset V_m$ is a non-zero invariant space, now consider $w \in W$. Then

$$w = b_0 z_1^m + b_1 z_1^{m-1} z_2 + \dots + b_m z_2^m$$

with at least one of the b_k's non-zero. Let k_0 be the smallest value for which b_k is non-zero. Then

$$\pi_m(E)\left(z_1^{m-k_0} z_2^{k_0}\right) = z_1 \partial_{z_2}\left(z_1^{m-k_0} z_2^{k_0}\right) = k_0 z_1^{m-k_0+1} z_2^{k_0-1}.$$

Since W was assumed to be invariant under π_m then we see that it must contain $z_1^{m-k_0+1} z_2^{k_0-1}$ so that the smallest non-zero b_k must in fact be b_0 (seen by repeatedly acting with $\pi_m(E)$). However if $b_0 \neq 0$ then if $(\pi_m(F))^n w$ is in W $\forall n \in \mathbb{Z}^+$ then we see that $W = V$. Hence π_m is irreducible.

Comments

- Notice that each representation π_m is labelled by one number m, or equivalently to the dimension of V_m ($\dim V_m = m+1$).
- $\pi_m(E)$ and $\pi_m(F)$ raise and lower the power of z_1 in each basis term.
- There are unique lowest and highest powers of z_1 as

$$\pi_m(E)\, z_1^m = (z_1 \partial_{z_2})\, z_1^m = 0$$

$$\pi_m(F)\, z_2^m = (z_2 \partial_{z_1})\, z_2^m = 0.$$

I.e. $\pi_m(E)$ (and $\pi_m(F)$) are raising (and lowering) operators whose repeated action on z_2^m (z_1^m) recreates all the basis elements in V_m.

- As the complexification of $su(2)$ representations to obtain $sl(2,\mathbb{C})$ representations is unique:

$$\pi_m(X) = \pi_m(X) + i\pi_m(X)$$

$sl(2,\mathbb{C})$ $\quad$ $su(2)$

(Yet more abuse of notation.)

Then $\exists$ irreducible representations of $su(2)$ of dimension $m+1$.

- As $\frac{SU(2)}{\mathbb{Z}_2} \cong SO(3)$ there exist corresponding $so(3)$ representations labelled by $\frac{m}{2}$, called the spin, $j = \frac{m}{2}$. Hence j may be integer (bosons) or half-integer (fermions).

E.g.

Spin, $j=\frac{m}{2}$.	Dim(V) = m+1.	
0.	1.	Trivial representation.
½.	2.	Standard or Fundamental.
1.	3	Adjoint representation.

Direct Sums and Tensor Products of Representations.

Given two representations π_1 and π_2 of a group G, on V_1 and V_2 respectively, one can form two important representations:

1. The Direct Sum.

$$\pi_1 \oplus \pi_2\,(g_1, g_2) \equiv \begin{pmatrix} \pi_1(g_1 g_2). & 0 \\ 0. & \pi_2(g_1 g_2). \end{pmatrix}$$

$$= \begin{pmatrix} \pi_1(g_1)\pi_1(g_2). & 0. \\ 0. & \pi_2(g_1)\pi_2(g_2). \end{pmatrix}$$

$$= \begin{pmatrix} \pi_1(g_1). & 0. \\ 0. & \pi_2(g_1). \end{pmatrix} \begin{pmatrix} \pi_1(g_2). & 0. \\ 0. & \pi_2(g_2). \end{pmatrix}$$

$$= (\pi_1 \oplus \pi_2)(g_1)\,(\pi_1 \oplus \pi_2)(g_2).$$

if V_n has basis $\{e_1, \ldots e_n\}$ and V_m has basis $\{f_1, \ldots f_m\}$ then $\pi_1 \oplus \pi_2$ is a representation on V_{n+m} with basis $\{e_1, \ldots, e_n, f_1, \ldots, f_m\}$.

The direct sum gives a block diagonal representation $\left(\begin{array}{c|c} \pi_1 & 0 \\ \hline 0 & \pi_2 \end{array}\right)\left(\begin{array}{c} V_1 \\ \hline V_2 \end{array}\right)$.

As the representation matrices are block-diagonal they have invariant subspaces,

i.e. $\pi_1 \oplus \pi_2 : V_1 \to V_1$ and $\pi_1 \oplus \pi_2 : V_2 \to V_2$

so both V_1 and V_2 are invariant subspaces of $V_{1\oplus 2}$.

Hence it is a reducible representation (i.e. it is not irreducible).

2. The Tensor Product.

$\pi_1 \otimes \pi_2 : G \to GL(V_1 \otimes V_2)$ such that $(\pi_1 \otimes \pi_2)(g) = \pi_1(g) \otimes \pi_2(g)$.

This is a homomorphism as

$$\begin{aligned} \pi_1 \otimes \pi_2 (g_1 g_2) &= \pi_1(g_1 g_2) \otimes \pi_2(g_1 g_2) \\ &= \pi_1(g_1)\pi_1(g_2) \otimes \pi_2(g_1)\pi_2(g_2) \\ &= (\pi_1 \otimes \pi_2)(g_1)(\pi_1 \otimes \pi_2)(g_2) \end{aligned}$$

The basis for $V_1 \otimes V_2$ is $\{ e_i \otimes e_j \mid i = 1, 2, \ldots n = \dim(V_1),\ j = 1, 2, \ldots, m = \dim(V_2) \}$

i.e. $\dim(V_1 \otimes V_2) = nm$.

Comments.

- Consider the endomorphisms from a vector space to itself, any $A \in End(V_1 \otimes V_2)$ can be written as

$$A = \left(\begin{array}{c|c} A_{V_1V_1} & A_{V_1V_2} \\ \hline A_{V_2V_1} & A_{V_2V_2} \end{array} \right)$$

where $A_{V_iV_j} : V_j \to V_i$. For a direct sum $A_{V_1V_2} = 0 = A_{V_2V_1}$ and note that

$$\dim(End(V_1)) = n^2,$$

$$\dim(End(V_2)) = m^2$$

while $\dim(End(V_1 \otimes V_2)) = (nm)^2$.

The direct sum representations cannot generate all the endomorphisms of $V_1 \otimes V_2$.

However the product representations do generate all the endomorphisms of $V_1 \otimes V_2$ as

$$\begin{aligned} Dim(End(V_1 \otimes V_2)) &= n^2 m^2 \\ &= Dim(End(V_1))\, Dim(End(V_2)). \end{aligned}$$

While the direct sum never gives an irreducible representation (having non-trivial invariant subspaces $V_1 \oplus 0 \cong V_1$ and $0 \oplus V_2 \cong V_2$), it is less straightforward with the tensor product to discover whether or not $\pi_1 \otimes \pi_2$ is irreducible.

- Frequently one is interested in decomposing tensor products into direct sums of irreducible representations:

$$V_1 \otimes V_2 = U_1 \oplus U_2 \oplus \dots \oplus U_n.$$

When this is possible (i.e. when all representation matrices can be written in block diagonal form ) the group/algebra is called semisimple.

To rewrite $\pi_1 \otimes \pi_2$ in block-diagonal form requires a change of basis on $V_1 \otimes V_2$ such that

$$T(\pi_1 \otimes \pi_2(g))T^{-1} = \hat{\pi}_1(g) \oplus \hat{\pi}_2(g) \oplus \dots \oplus \hat{\pi}_n(g).$$

where $T \in \mathrm{End}(V_1 \otimes V_2)$.

The decomposition

$$\pi(G) \otimes \pi(G) = \sum_i a_i \hat{\pi}_i(G)$$

is called the Clebsch-Gordan decomposition.

Of course it is generally not possible to do this but an important example where it is possible is central to quantum mechanics and involves the representations of $SO(3)$ of spin j, which we will denote $[j]$. Recall these representations are isomorphic to the $su(2)$ representations of dimension $m+1$ where $j = \frac{m}{2}$.

The Clebsch-Gordan decomposition in this case is:

$$[j_1] \otimes [j_2] = [j_1+j_2] \otimes [j_1+j_2-1] \otimes [j_1+j_2-2] \otimes \dots \otimes [|j_1-j_2|]$$

SU(2) dimension: $(2j_1+1)\times(2j_2+1)$ $\quad 2(j_1+j_2)+1$ $\quad 2(j_1+j_2-1)+1$ $\quad \dots$ $\quad 2|j_1-j_2|+1$.

Simple examples are:

$$[0] \otimes [j] = [j]$$

$$1 \times 2j+1 = 2j+1$$

$$[\tfrac{1}{2}] \otimes [j] = [\tfrac{1}{2}+j] \otimes [-\tfrac{1}{2}+j] \qquad \text{where } j \geq \tfrac{1}{2}.$$

$$2 \times 2j+1 = 2+2j \times 2j$$

$[\frac{1}{2}]$ which is the 2-dimensional representation of $su(2)$ is called the fundamental representation as its tensor products with itself generate all other irreducible representations of $su(2)$:

$$[\tfrac{1}{2}] \otimes [\tfrac{1}{2}] = [1] \otimes [0]$$
$$3 \qquad 1$$

$$[1] \otimes [\tfrac{1}{2}] = [\tfrac{3}{2}] \otimes [\tfrac{1}{2}]$$
$$4 \qquad 2$$

Towards a Classification of the Lie Algebras.

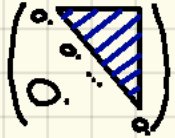
Having a description of a Lie algebra as a matrix representation there are a number of questions from linear algebra which we may ask about the matrices. We will postpone a discussion of eigenvalues for the moment and focus on the standard forms a matrix can be reduced to by a change of basis.

Focussing on the adjoint representation we try to obtain matrices of the form:

1. $\begin{pmatrix} & 0 & \\ & & \end{pmatrix}$ corresponds to an abelian algebra

$$[Z, X_i] = 0 \qquad \forall\ Z, X_i \in \mathcal{G}$$

The zero matrix

2. 

These matrices are nilpotent: there exists an integer $n > 0$ such that $(\pi(X))^n = 0$, the zero matrix.

Upper triangular matrices.

$$[Z, X_i] = \pi(Z)_i^{\ j} X_j \subset \text{span}\{X_{i+1}, X_{i+2}, \dots X_n\}$$

↑ non-zero only for $j > i$

3. 

Upper triangle with some non-zero entries on the diagonal.

Such an algebra is said to be solvable.

$$[Z, X_i] = \pi(Z)_i^{\ j} X_j \subset \text{span}\{X_i, X_{i+1}, \dots X_n\}$$

↑ non-zero only for $j \geq i$

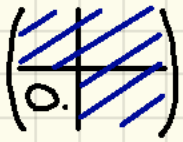
Let $V_i \equiv \text{span}\{X_i, \dots, X_n\}$ so that $\mathfrak{g} = V_1$

then each V_i is mapped onto itself by ad_Z with

$$V_1 \supset V_2 \supset V_3 \dots \supset V_n$$

all being invariant subalgebras.

Every nilpotent algebra is solvable.

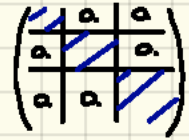
4. 

Reducible

This is reducible as $\left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} Av_1 + Bv_2 \\ Cv_2 \end{pmatrix}$ so

$\begin{pmatrix} v_1 \\ 0 \end{pmatrix} \to \begin{pmatrix} v_1' \\ 0 \end{pmatrix}$ and V_2 is an invariant subspace $\Rightarrow$ reducible.

Such algebras are sometimes called nonseminimple.

5. 

Fully-reducible.

This representation can be fully reduced into a direct sum of representations.

Such algebras are called **semisimple.**

e.g. $\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} Av_1 \\ Bv_2 \end{pmatrix}$

or $\mathrm{ad}|_{V_1}: \begin{array}{l} V_1 \to V_1 \\ V_2 \to 0 \end{array}$ $\mathrm{ad}|_{V_2}: \begin{array}{l} V_1 \to 0 \\ V_2 \to V_2. \end{array}$

Hence V_1 and V_2 may be studied independently.

A semisimple Lie algebra reproduces itself under the adjoint action.

6. 

Irreducible.

The representation is irreducible.

Such algebras are called **simple.**

Every generator X_k can be written as a commutator of X_i and X_j:

$$X_k = [X_i, X_j].$$

A simple Lie algebra reproduces itself under the adjoint action.

The Cartan-Killing Form.

The Lie algebra spans a vector space (the tangent space) at the group identity.

We can endow this space with an inner product and if we represent the algebra by matrices in the adjoint representation we need an inner product on matrices (with complex entries in general).

$\Rightarrow \quad (A, B) = \mathrm{Tr}(A^\dagger B)$

The Frobenius inner product (A,B are matrices)
The Hilbert-Schmidt inner product.
(A,B are operators).

This is a positive (non-negative) inner product.

The Cartan-Killing Form is a similar but indefinite inner product:

defined on the adjoint representation of $\mathfrak{g}$, i.e.

$$(X_i, X_j) = \mathrm{Tr}(\mathrm{ad}_{X_i}, \mathrm{ad}_{X_j}) = \mathrm{Tr}(\pi(X_i)\pi(X_j)).$$

Example: the Cartan-Killing Form on $sl(2,\mathbb{R})$.

Recall that $\pi(aH+bE+cF) = \begin{pmatrix} 0 & -c & b \\ -2b & 2a & 0 \\ 2c & 0 & -2a \end{pmatrix}$ for the adjoint.

Then $(aH+bE+cF, aH+bE+cF) = Tr \begin{pmatrix} 0 & -c & b \\ -2b & 2a & 0 \\ 2c & 0 & -2a \end{pmatrix} \begin{pmatrix} 0 & -c & b \\ -2b & 2a & 0 \\ 2c & 0 & -2a \end{pmatrix}$

$$= Tr \begin{pmatrix} 4bc & -2ac & -2ab \\ -4ab & 2bc+4a^2 & -2b^2 \\ -4ac & -2c^2 & 2cb+4a^2 \end{pmatrix}$$

$$= 8a^2 + 8bc.$$

The Cartan-Killing form splits the adjoint representation into three subspaces:

$$\mathfrak{g} = V_- + V_0 + V_+$$

$(x,x)<0 \quad (x,x)=0 \quad (x,x)>0.$

In fact V_- consists of compact operators.

V_0 consists of nilpotent elements.

V_+ consists of non-compact operators.

For example, consider $\pi(x)$ nilpotent i.e. $\pi(x)_i^{\ j} \neq 0$ for $i>j$ then

$$\begin{aligned} Tr(\pi(x)\pi(x)) &= Tr\Big(\sum_k \pi(x)_i^{\ k}\, \pi(x)_k^{\ j}\Big) \\ &= Tr\Big(\sum_{k=i+1} \pi(x)_i^{\ k}\, \pi(x)_k^{\ j}\Big) \\ &= \sum_{k=i+1} \pi(x)_i^{\ k}\, \underbrace{\pi(x)_k^{\ i}}_{=0} \\ &= 0. \end{aligned}$$

Consider the generators $(E-F) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $(E+F) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ within $sl(2,\mathbb{R})$.

Now $e^{\theta(E-F)} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ which is a compact transformation.

(as all matrix entries are bounded)

$e^{\theta(E+F)} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$ which is a non-compact transformation.

Then $(E-F, E-F) = Tr \begin{pmatrix} 0 & 1 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}$

$= Tr \begin{pmatrix} -4 & & \\ & -2 & \\ & & -2 \end{pmatrix}$

$= -8$ //

$(E+F, E+F) = 8$ //

We can use the Cartan-Killing form on a Lie algebra to identify the maximal nilpotent sub-algebra thereby splitting:

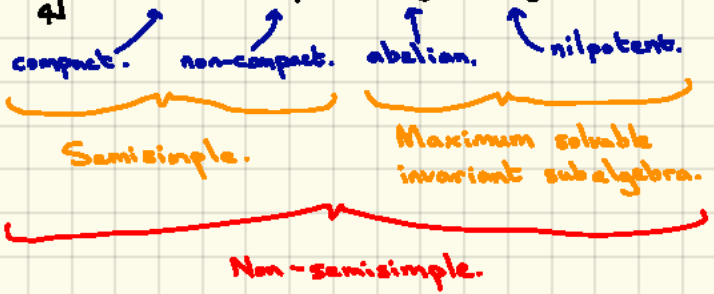
$$\mathfrak{g} = V_0 + (V_+ + V_-)$$

Now removing V_0 and constructing the adjoint representation for the remaining generators, $\mathfrak{g}'$ and use the Cartan-Killing form to split $\mathfrak{g}'$ into

$$\mathfrak{g}' = V_0' + (V_+' + V_-').$$

Then we have,

$$\mathfrak{g}' = V_-' + V_+' + V_0' + V_0$$



Example: $sl(2,\mathbb{R})$

$$\pi(aH+bE+cF) = \begin{pmatrix} 0. & -c. & b. \\ -2b. & 2a. & 0. \\ 2c. & 0. & -2a. \end{pmatrix}$$

$$(aH+bE+cF,\ aH+bE+cF) = 8a^2+8bc$$

There is no nilpotent subalgebra $\Rightarrow$ $sl(2,\mathbb{R})$ is simple or semisimple.

> If the Cartan-Killing form is non-degenerate then the algebra is semisimple.

Example: Algebra for the rotations and translations of $\mathbb{R}^2$.

$$X = \begin{pmatrix} 0 & -\theta & a \\ \theta & 0 & b \\ 0 & 0 & 0 \end{pmatrix} = \theta R + a T_x + b T_y.$$

$$[R, T_x] = \left[\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = T_y$$

$$[R, T_y] = -T_x$$

$$[T_x, T_y] = 0.$$

$$\therefore \pi(\theta R + a T_x + b T_y) = \begin{pmatrix} 0 & b & -a \\ 0 & 0 & \theta \\ 0 & -\theta & 0 \end{pmatrix}$$

$$\therefore \quad (\theta R + a T_x + b T_y, \theta R + a T_x + b T_y) = \mathrm{Tr} \begin{pmatrix} 0 & -\theta^2 & \\ & -\theta^2 & 0 \end{pmatrix} = -2\theta^2$$

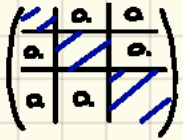
So $a T_x + b T_y$ is a nilpotent subalgebra spanning V_0, then $\mathfrak{g}' = \theta R$

For which $\pi'(R) = 0$ as $[R, R] = 0$.

$\therefore V_0'$ is spanned by R.

So $\mathfrak{g} = V_0' + V_0$ is a soluble algebra.

Summary.

- A representation of a group G on V, $\Pi: G \to GL(V)$ is a homomorphism.
- A representation of a Lie algebra $\mathfrak{g}$ on V, $\pi: \mathfrak{g} \to GL(V)$ is a Lie algebra homomorphism, i.e. $\pi([X,Y]) = [\pi(X), \pi(Y)]$.
- The adjoint representation is given by $[Z, X_i] = \underbrace{\pi(Z)_i^{\ j}}_{\text{adjoint.}} X_j$
- There is an irreducible representation of $\mathfrak{sl}(2,\mathbb{F})$ for $\dim(V) = m+1 \ \forall\, m \in \mathbb{Z}^+$.
-  solvable algebra  (Fully reducible) semisimple algebra  Irreducible. simple algebra
- $(X_i, X_j) = Tr(ad_{X_i}, ad_{X_j})$ is the Cartan-Killing Form.

Lecture IV: Eigenvalues.

- $\mathfrak{sl}(2,\mathbb{C})$ again
- $\mathfrak{sl}(3,\mathbb{C})$
- Weights and Roots
- Dynkin diagrams and Cartan Matrices.

Eigenvectors of the adjoint Representation.

An eigenoperator X for Z with eigenvalue λ satisfies

$$[Z, X] = \lambda X.$$

Let us try to find a set of basis vectors $X = \sum_i a_i X_i$ which satisfy the relation above.

$$[Z, X] = [Z, \sum_i a_i X_i] = \sum_i a_i \pi(Z)_i{}^j X_j = \lambda \sum_i a_i X_i$$

$$\Rightarrow \sum_{i,j} a_i \left(\pi(Z)_i{}^j - \lambda \mathbb{1}_i{}^j\right) X_j = 0$$

$$\Rightarrow \det\left(\pi(Z) - \lambda \mathbb{1}\right) = 0. \quad \text{(or all } a_i = 0\text{)}.$$

This gives a polynomial equation in λ and a_i having the form

$$\sum_k \lambda^{n-k} p_k(a_i) = 0$$

where $p_k(a_i)$ is a polynomial in a_i of order k.

Our aim is to find λ as functions of a_i and then use the $\sum a_i X_i$ as a basis to diagonalize $\pi(Z)$

Example: $sl(2,\mathbb{C})$

$$\pi(z) = \pi(aH + bE + cF) = \begin{pmatrix} 0 & -c & b \\ -2b & 2a & 0 \\ 2c & 0 & -2a \end{pmatrix}$$

$$\therefore \quad \det(\pi(z) - \lambda \mathbb{1}) = \begin{pmatrix} -\lambda & -c & b \\ -2b & 2a-\lambda & 0 \\ 2c & 0 & -2a-\lambda \end{pmatrix}$$

$$= -\lambda(2a-\lambda)(-2a-\lambda) + 2b(-c(-2a-\lambda)) + b(-2c(2a-\lambda))$$

$$= -\lambda^3 + \lambda(4a^2 + 4cb)$$

$$\equiv \lambda^3 p_0(a,b,c) + \lambda p_1(a,b,c)$$

(i.e. $p_1(a,b,c) = 4a^2 - 4cb$ $\quad p_0(a,b,c) = -1$ $\quad p_2 = p_3 = 0$.) $\rightsquigarrow$

replace a_i with X_i in $p_1(a_i)$ to construct elements which commute with $\mathfrak{g}$ called Casimirs.

$$= \lambda(-\lambda^2 + 4a^2 + 4cb)$$

$$= \lambda(-\lambda + 2\sqrt{a^2+cb})(\lambda + 2\sqrt{a^2+cb})$$

$\det = 0$

$\Rightarrow \lambda = 0, \lambda = \pm n$ where $n = 2\sqrt{a^2+cb}$ which exists over $\mathbb{C}$.

When $\lambda = 0$ then $[Z, X] = 0 \Rightarrow X = Z$.

When $\lambda = \pm 2\sqrt{a^2 + bc}$ we may take $a = 1, b = 0, c = 0$ so that $Z = H$ giving:

$$\pi(H) = \begin{pmatrix} 0. & 0. & 0. \\ 0. & 2. & 0. \\ 0. & 0. & -2. \end{pmatrix}$$

and $[\pi(H), \pi(E)] = \left[\begin{pmatrix} 0. & 0. & 0. \\ 0. & 2. & 0. \\ 0. & 0. & -2. \end{pmatrix}, \begin{pmatrix} 0. & 0. & 1. \\ -2. & 0. & 0. \\ 0. & 0. & 0. \end{pmatrix} \right]$

$$= \begin{pmatrix} 0. & 0. & 2. \\ -4. & 0. & 0. \\ 0. & 0. & 0. \end{pmatrix}$$

$$= 2\pi(E)$$

$$[\pi(H), \pi(F)] = -2\pi(F).$$

$$[\pi(H), \pi(H)] = 0.$$

$SL(2, \mathbb{R})$ was simple enough that we understood the above relations, but this scheme generalizes.

The Irreducible Representations of $sl(2,\mathbb{C})$ ($\Leftrightarrow$ $su(2)$ & $so(3)$ irreps).

Theorem: For each integer $m \geq 0$ $\exists$ an irreducible representation of $sl(2,\mathbb{C})$ acting on a finite-dimensional, complex vector space V.

Let π be such an irrep of $sl(2,\mathbb{C})$ acting on V, where $\dim V$ is finite.

We cannot assume $\pi(H)$ is diagonalizable but because we are working over $\mathbb{C}$ we know an eigenvalue $\alpha \in \mathbb{C}$ exists with eigenvector v:

$$\pi(H)v = \alpha v.$$

Then
$$\pi(H)\pi(E)v = [\pi(H), \pi(E)]v + \pi(E)\pi(H)v$$
$$= \pi([H,E])v + \pi(E)(\alpha v)$$
$$= (\alpha+2)\pi(E)v$$

$\Rightarrow$ either $\pi(E)v = 0$ or is an eigenvector for $\pi(H)$ with eigenvalue $(\alpha+2)$.

Similarly, $\pi(H)\pi(F)v = (\alpha-2)\pi(F)v$

and either $\pi(F)v = 0$ or $\pi(F)v$ is an eigenvector of $\pi(H)$ with eigenvalue $(\alpha-2)$.

$\pi(E)$ and $\pi(F)$ are called the raising and lowering operators respectively.

So, by repetition, we have

$$\pi(H)(\pi(E))^n v = (\alpha+2n)(\pi(E))^n v \qquad \forall\ n \in \mathbb{Z}^+.$$

But this creates an infinite set of eigenvectors for $\pi(H)$ all with distinct eigenvalues.

Now an operator on a finite-dimensional vector space V can only have a finite number of distinct eigenvalues ($\det(\pi(H) - \lambda\mathbb{1}) = 0$ is a polynomial of degree $\dim V$ over $\mathbb{C}$).

So there exists some $n=N+1$ such that

$$\pi(E)^{N+1} v = 0$$

Let $\pi(E)^N v \equiv v_0$ and $\alpha+2N \equiv \lambda$ then

$$\pi(H)v_0 = \lambda v_0 \qquad \text{with} \quad \pi(E)v_0 = 0.$$

$$\pi(H)\pi(F)^k v_0 = (\lambda - 2k)\pi(F)^k v_0 \qquad k \in \mathbb{Z}^+$$

and again this string of eigenvectors must terminate for some k.

Lemma: Let $v_k = \pi(F)^k v_0$ and so $\pi(H)v_k = (\lambda - 2k)v_k$ then

$$\pi(E)v_k = (k\lambda - k(k-1))v_{k-1} \qquad (k > 0).$$

Proof (by induction)

Assume $\pi(E)v_k = (k\lambda - k(k-1))v_{k-1}$ to show $\pi(E)v_{k+1} = ((k+1)\lambda - (k+1)k)v_k$:

$$\pi(E)v_{k+1} = \pi(E)\pi(F)v_k$$

$$= \pi([E,F])v_k + \pi(F)\pi(E)v_k \qquad \text{Assume inductive step.}$$

$$= \pi(H)v_k + \pi(F)\left(k\lambda - k(k-1)\right]v_{k-1}$$

$$= (\lambda - 2k + k\lambda - k^2 + k)v_k$$

$$= ((k+1)\lambda - k(k+1))v_k \qquad \text{as required.}$$

The base step is to prove the relation for $k=1$:

$$\pi(E)\,v_1 = \pi(E)\pi(F)\,v_0 = \pi([E,F])\,v_0 + \pi(F)\pi(E)\,v_0$$

$$= \pi(H)\,v_0$$

$$= \lambda v_0. \qquad \text{As required.}$$

Now we have $\pi(E)\,v_k = \big(k\lambda - k(k-1)\big)\,v_{k-1}$

We have observed that $v_k = 0$ for some $k = M+1$ i.e. $v_{M+1} = 0$.

So $\pi(E)\,v_{M+1} = 0$

$$= \big((M+1)\lambda - (M+1)M\big)\,v_M$$

but $v_M \neq 0$ so $\lambda = M$.

We have a finite string of $\pi(H)$ eigenvalues and eigenvectors:

$v_0 \quad v_1 \quad v_2 \quad \cdots \quad v_{M=\lambda}$

$\lambda \quad \lambda-2 \quad \lambda-4 \quad \lambda-2\lambda=-\lambda$

} which span a vector space of dimension $M+1$. which is invariant under $\pi(H)$, $\pi(E)$ and $\pi(F)$. As π is irreducible this is all of V.

- This construction should be compared with that action of $\mathfrak{sl}(2,\mathbb{C})$ on $\sum_i a_i z_1^{m-i} z_2^i$ constructed earlier. They are equivalent.

- As $\dim V = m+1$ is an integer then all eigenvalues of $\pi(H)$ must be integers.

$sl(3,\mathbb{C})$

This is the complexification of both $sl(3,\mathbb{R})$ and $su(3)$.

It consists of the traceless matrices in $M_3(\mathbb{R})$ (the algebra is complex linear).

We can make a smart choice of basis:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$E_{12} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

N.B. $Tr(H_i) = 0$, $F_i = (E_i)^T$ and $[E_1, E_2] = E_{12}$ while $[F_2, F_1] = F_{12}$.

We have $[H_1, E_1] = 2E_1$ $\quad$ $[H_2, E_2] = 2E_2$

$[H_1, F_2] = F_2$ $\quad$ $[H_2, F_1] = F_1$

$[H_1, H_2] = 0$ $\quad$ $[H_2, H_1] = 0$

$[H_1, E_2] = -E_2$ $\quad$ $[H_2, E_1] = -E_1$

$[H_1, F_1] = -2F_1$ $\quad$ $[H_2, F_2] = -2F_2$.

n.b. E_1, E_2, F_1, F_2 are a basis for the nilpotent part of $\mathfrak{sl}(3, \mathbb{C})$ which are all eigenoperators for H_1 and H_2 (with unique eigenvalues).

n.b. By the Jacobi identity we have

$$[H_1, [E_1, E_2]] + [E_1, [E_2, H_1]] + [E_2, [H_1, E_1]] = 0$$

$$[H_1, [E_1, E_2]] + [E_1, E_2] + [E_2, 2E_1] = 0 \quad \Rightarrow [H_1, E_{12}] = E_{12} \quad \text{etc.}$$

In the following we take π to stand for a general representation of $sl(3,\mathbb{C})$, and not just the adjoint.

Definition: If π is a representation of $sl(3,\mathbb{C})$ on V then the ordered pair $\mu = (m_1, m_2)$ is called **a weight** for π if $\exists\ v \neq 0$ in V such that

$$\pi(H_1)v = m_1 v$$

$$\pi(H_2)v = m_2 v.$$

The vector v is called **a weight vector**. The space spanned by all the vectors v satisfying the above eigenvector relations is called **a weight space**. The **multiplicity** of a weight is the dimension of the corresponding weight space.

Comments.

- All representations of $\mathfrak{sl}(3,\mathbb{C})$ have at least one weight, as over $\mathbb{C}$ $\pi(H_1)$ has an eigenvalue m_1 (which may be complex). Let $V_1 \subset V$ be the eigenspace for which $v \in V_1$ satisfy $\pi(H_1)v = m_1 v$. Now as

$$[H_1, H_2] = 0 \quad \text{then} \quad [\pi(H_1), \pi(H_2)] = 0$$

hence $$\pi(H_1)\pi(H_2)v = \pi(H_2)\pi(H_1)v = m_1 \pi(H_2)v$$

$$\Rightarrow \pi(H_2)v \in V_1$$

i.e. $\pi(H_2): V_1 \to V_1$ and as we are working over $\mathbb{C}$ an eigenvalue for $\pi(H_2)$ exists, $m_2 \in \mathbb{C}$. So v is an eigenvector for both $\pi(H_1)$ and $\pi(H_2)$ with weight (m_1, m_2).

- When π is the adjoint representation the weight is called **a root** $\alpha = (a_1, a_2)$ if a_1 and a_2 are both non-zero and $\exists$ non-zero $X \in \mathfrak{g}$ such that $[H_i, X] = a_i X$

From the commutation relations:

$[H_1, E_1] = 2E_1$ $\quad [H_2, E_1] = -E_1$

$[H_1, F_2] = F_2$ $\quad [H_2, F_2] = -2F_2$

$[H_1, H_2] = 0$ $\quad [H_2, H_1] = 0$

$[H_1, E_2] = -E_2$ $\quad [H_2, E_2] = 2E_2$

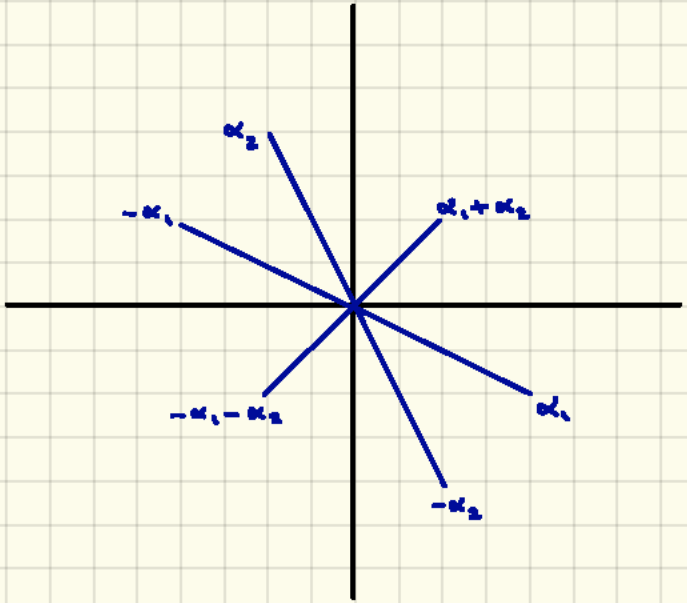
$[H_1, F_1] = -2F_1$ $\quad [H_2, F_1] = F_1$

$[H_1, E_{12}] = E_{12}$ $\quad [H_2, E_{12}] = E_{12}$

$[H_1, F_{12}] = -F_{12}$ $\quad [H_2, F_{12}] = -F_{12}$

Algebra element X	Root α
E_1	$(2, -1)$
F_2	$(1, -2)$
H_1, H_2	no root
E_2	$(-1, 2)$
F_1	$(-2, 1)$
E_{12}	$(1, 1)$
F_{12}	$(-1, -1)$

So $sl(3, \mathbb{C})$ has 6 roots.



E_1 $(2,-1) \equiv \alpha_1$

F_2 $(1,-2)$

E_2 $(-1,2) \equiv \alpha_2$

F_1 $(-2,1)$

E_{12} $(1,1)$

F_{12} $(-1,-1)$

We can make a choice of basis roots such that all roots are positive sums of α_1 and α_2 or all negative sums of α_1 and α_2. Such roots are called the

simple positive roots.

Lemma: Let $\alpha = (a_1, a_2)$ be a root and X_α the corresponding root vector in $sl(3,\mathbb{C})$. Let π be a representation of $sl(3,\mathbb{C})$ with weight $\mu = (m_1, m_2)$ and let $v \neq 0$ be a corresponding weight vector. Then

$$\pi(H_i)\pi(X_\alpha)v = (m_i + a_i)\pi(X_\alpha)v \qquad \text{for } i = 1, 2.$$

Hence either $\pi(X_\alpha)v = 0$ or $\pi(X_\alpha)v$ is a new weight vector with weight $(m_1 + a_1, m_2 + a_2) = \mu + \alpha$.

Proof.

$$\begin{aligned}
\pi(H_i)\pi(X_\alpha)v &= [\pi(H_i), \pi(X_\alpha)]v + \pi(X_\alpha)\pi(H_i)v \\
&= \pi([H_i, X_\alpha])v + \pi(X_\alpha)(m_i v) \\
&= \pi(a_i X_\alpha)v + \pi(X_\alpha)(m_i v) \\
&= (a_i + m_i)\pi(X_\alpha)v.
\end{aligned}$$

Comments

- $\pi(X_\alpha)$ raises or lowers a weight by α.

We can be more precise as

$$\pi(H_1)\pi(E_1)v = (2+m_1)\pi(E_1)v \qquad \pi(H_1)\pi(F_1)v = (-2+m_1)\pi(F_1)v$$

↑ Recall $\alpha_1 \in (2,-1)$

$$\pi(H_2)\pi(E_1)v = (-1+m_2)\pi(E_1)v \qquad \pi(H_2)\pi(F_1)v = (1+m_2)\pi(F_1)v$$

So $\pi(E_1): \mu \longrightarrow \mu+\alpha_1$ $\qquad \pi(F_1): \mu \longrightarrow \mu-\alpha_1$

and $\pi(E_2): \mu \longrightarrow \mu+\alpha_2$ $\qquad \pi(F_2): \mu \longrightarrow \mu-\alpha_2$

$\pi(E_i)$ raise the weight by α_i while $\pi(F_i)$ lower the weight by α_i

- Given a weight μ one constructs a string of weights if a weight vector exists, but of course some of the weight vectors $\pi(X_\alpha)v$ will be zero.

This is the case if V is a finite-dimensional representation.

This is similar to the representations of $sl(2,\mathbb{C})$ where there was a unique highest (or lowest) vector which was annihilated by $\pi(E)$ (or $\pi(F)$).

The ordering of the terms $z_1^{m-j}z_2^j$ by the exponent of j was natural.

For our $sl(3,\mathbb{C})$ example every weight is of the form

$$\mu_i = \mu + a_i\alpha_1 + b_i\alpha_2$$

Hence we can set a partial order on the weights as follows:

$$\mu_i \geq \mu_j \quad \text{if} \quad \mu_i - \mu_j = (a_i - a_j)\alpha_1 + (b_i - b_j)\alpha_2 \quad \text{for } (a_i - a_j) \geq 0 \text{ and } (b_i - b_j) \geq 0.$$

Example: The (1,0) weight of $sl(3,\mathbb{C})$

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We seek a vector in $\mathbb{R}^3$ such that $H_1 v = v \qquad H_2 v = 0$

i.e. let $v = x\underline{e}_1 + y\underline{e}_2 + z\underline{e}_3$ then

$$H_1 v = x\underline{e}_1 + y\underline{e}_2 + z\underline{e}_3 = x\underline{e}_1 - y\underline{e}_2 \quad \Rightarrow \quad y = 0, \; z = 0$$

Then if $v = x\underline{e}_1$ we have

$$H_2 v = 0$$

So $v = \underline{e}_1$ is a weight vector with weight $(1,0)$. We will denote it $v_{(1,0)} = \underline{e}_1$.

Now $$E_1 v_{(1,0)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0 \qquad E_2 v_{(1,0)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$$

So $v_{(1,0)}$ is a highest weight. The lower weights are

$$F_1 v_{(1,0)} = \begin{pmatrix} 0 & * & * \\ 1 & * & * \\ 0 & * & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \underline{e}_2$$

On which we have

$$H_1 \underline{e}_2 = \begin{pmatrix} 1 & * & * \\ 0 & -1 & * \\ 0 & 0 & * \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -\underline{e}_2 \qquad H_2 \underline{e}_2 = \begin{pmatrix} * & * & * \\ 0 & 1 & * \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \underline{e}_2$$

So $\underline{e}_2$ is a weight vector with weight $(-1,1) = (1,0) - \alpha_1 = (1,0) - (2,-1)$ we denote it $v_{(-1,1)} = \underline{e}_2$.

Now $F_1 v_{(-1,1)} = 0$, but $F_2 v_{(-1,1)} = \begin{pmatrix} 0 & * & * \\ 0 & * & * \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \underline{e}_3$.

And $\underline{e}_3$ is a weight vector with weight $(1,0) - \alpha_1 - \alpha_2 = (-1,1) - (-1,2) = (0,-1)$, so $v_{(0,-1)} = \underline{e}_3$.

This is the lowest weight in this 3-dimensional representation of $\mathfrak{sl}(3,\mathbb{F})$.

Root Spaces and Inner Products.

The number of independent roots is the rank of the Lie algebra.

The roots can be collected into a root vector living in a root lattice.

If α and β are two roots associated with subspaces V_α and V_β

i.e. $[H_i, X_\alpha] = \alpha_i X_\alpha \qquad [H_i, X_\beta] = \beta_i X_\beta$

with X_α and X_β in $\mathfrak{g}$ then

$$[H_i,[X_\alpha,X_\beta]] + [X_\alpha,[X_\beta,H_i]] + [X_\beta,[H_i,X_\alpha]] = 0$$

$$[H_i,[X_\alpha,X_\beta]] + [X_\alpha,-\beta_i X_\beta] + [X_\beta,\alpha_i X_\alpha] = 0$$

$$\therefore [H_i,[X_\alpha,X_\beta]] = (\beta_i + \alpha_i)[X_\alpha,X_\beta].$$

Or $[V_\alpha, V_\beta] \subset V_{\alpha+\beta}$.

N.B. Let us collect together the components of a root vector by writing $[H, E_\alpha] = \alpha E_\alpha = \alpha(H) E_\alpha$

i.e. H is a vector of H_i.

The commutator $[X_\alpha, X_{-\alpha}]$ satisfies

$$[H,[X_\alpha, X_{-\alpha}]] + [X_\alpha,[X_{-\alpha}, H]] + [X_{-\alpha},[H, X_\alpha]] = 0$$

$$[H,[X_\alpha, X_{-\alpha}]] + [X_\alpha, -X_{-\alpha}] + [X_{-\alpha}, \alpha X_\alpha] = 0$$

So $[X_\alpha, X_{-\alpha}]$ is in the Cartan subalgebra, i.e.

$$[X_\alpha, X_{-\alpha}] = \vec{\alpha}^i H_i$$

Now an inner-product can be introduced between $\vec{\alpha}^i$ (co-roots) via the Cartan-Killing form:

$$\langle \alpha, \beta \rangle = (\vec{\alpha}^i H_i, \vec{\alpha}^j H_j)$$

E.g. For $sl(3,\mathbb{C})$ $\quad [E_1, F_1] = H_1 \Rightarrow \vec{\alpha}_1 = (1,0)$ i.e. $\alpha_1(H) = \vec{\alpha}^i H_i = H_1$

$[E_2, F_2] = H_2 \Rightarrow \vec{\alpha}_2 = (0,1)$ i.e. $\alpha_2(H) = H_2$

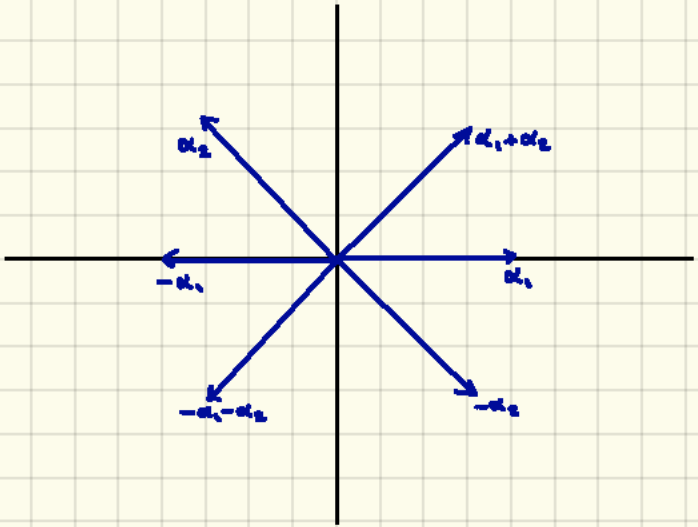
$[E_{\alpha_3}, F_{\alpha_3}] = H_1 + H_2 \Rightarrow \vec{\alpha}_{\alpha_3} = (1,1) = \vec{\alpha}_1 + \vec{\alpha}_2 \quad \alpha_{\alpha_3}(H) = H_1 + H_2$.

Then $\langle \alpha_1, \alpha_1 \rangle = \mathrm{Tr}\begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix} = 2$, $\langle \alpha_2, \alpha_2 \rangle = \mathrm{Tr}\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} = 2$, $\langle \alpha_1, \alpha_2 \rangle = \mathrm{Tr}\begin{pmatrix} * & 0 \\ 0 & -1 \end{pmatrix} = -1$.

The roots of $sl(3,\mathbb{C})$ all have length-squared 2.

$\langle \alpha_1, \alpha_2 \rangle = -1 = |\alpha_1||\alpha_2| \cos\theta_{12} \quad \Rightarrow \cos\theta_{12} = \frac{-1}{2} \quad \Rightarrow \theta_{12} = 2\pi/3.$

Giving a symmetric structure to the $sl(3,\mathbb{C})$ root space:



There are three $sl(2,\mathbb{C})$ root systems here $\pm\alpha_1$, $\pm\alpha_2$ and $\pm(\alpha_1+\alpha_2)$.

N.B. $\alpha(H_i) = 2\frac{\langle \alpha, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}$ $[H_i, X_j] = 2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} X_j \equiv A_{ij} X_j$ i.e. $[H_1, E_1] = 2E_1$, $[H_1, E_2] = -E_2$ etc.

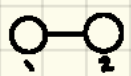
Dynkin diagrams, Cartan Matrices and Abstract Root Systems.

The semisimple Lie algebras are classified following the identification of $sl(2,\mathbb{C})$ subalgebras, each one containing a H_i, E_i and F_i generator.

Each of these contains a one-dimensional root system containing $\pm\alpha_i$.

Against the diagonalised H_i roots and root vectors for the semisimple Lie algebras are defined - from here the representation theory may be developed for each algebra.

The end result is described in terms of a Dynkin diagram which represents an $sl(2,\mathbb{C})$ sub-algebra for each positive simple root of the algebra by a node ○.

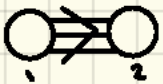
So $sl(2,\mathbb{C})$ is just ○ while $sl(3,\mathbb{C})$ is 

.

The Dynkin diagram encodes the Cartan matrix $A_{ij} \equiv \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}$ a matrix of root inner products as follows:

- $A_{ij} = 2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}$
- the rank of A is the number of nodes in the Dynkin diagram
- $A_{ii} = 2$
- $A_{ij}A_{ji}$ = the number of lines joining node i to node j
- $A_{ij} \in \{0, -1, -2, -3\}$
- if n lines join node i to node j and an arrowhead points from node i to node j then $|\alpha_i|^2 = n|\alpha_j|^2$ which determines A_{ij} and A_{ji}.

Example: G_2

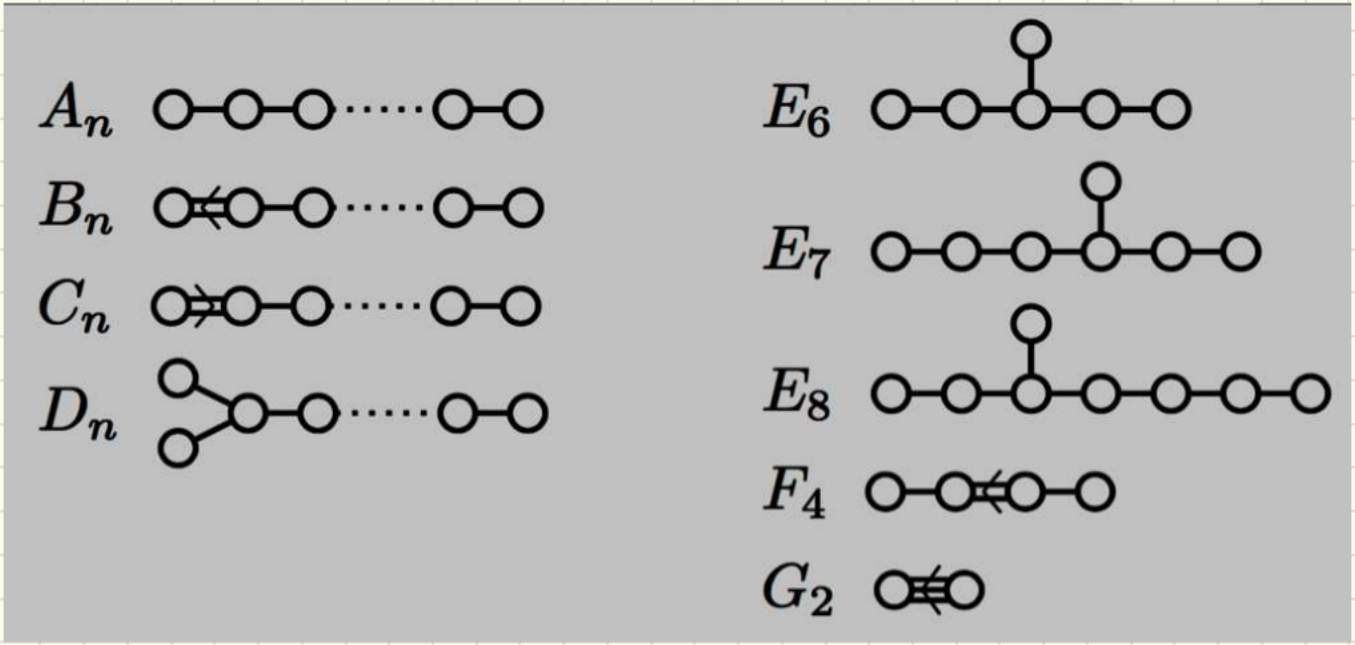


$A_{12}A_{21} = 3 \Rightarrow A_1, A_2 \in \{-1, -3\}$

$\frac{A_{12}}{A_{21}} = \frac{\langle \alpha_1, \alpha_2 \rangle}{\langle \alpha_1, \alpha_2 \rangle} = \frac{3}{1}$ as $|\alpha_1|^2 = 3|\alpha_2|^2$

$\therefore A = \begin{pmatrix} 2. & -3. \\ -1. & 2. \end{pmatrix}$

Dynkin Diagrams



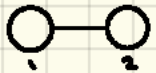
$A_n \cong sl(n+1, \mathbb{C})$

$B_n \cong so(2n+1)$

$C_n \cong sp(2n)$

$D_n \cong so(2n)$

Example: $sl(3,\mathbb{C})$



$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ $\Rightarrow \langle \alpha_1, \alpha_1 \rangle = 2 = \langle \alpha_2, \alpha_2 \rangle$, $\langle \alpha_1, \alpha_2 \rangle = -1$

Generators associated with α_1 and α_2 are H_1, E_1, F_1

H_2, E_2, F_2

Now close under the adjoint action:

$[H_1, E_1] = \langle \alpha_1, \alpha \rangle E_1 = 2E_1$ $\quad [H_1, E_2] = -E_2$

$[H_2, E_1] = -E_1$ $\quad [H_2, E_2] = 2E_2$

By the Jacobi identity,

$[H_1, [E_1, E_2]] = E_{12}$ if $\alpha_1 + \alpha_2$ is not zero.

$[H_1, [E_1, E_{12}]] + [E_1, [E_{12}, H_1]] + [E_{12}, [H_1, E_1]] = 0$

$(4-2)[E_1, E_{12}] - [E_1, E_{12}] + [E_{12}, 2E_1] = -[E_1, E_{12}] = 0.$ $\Rightarrow 2\alpha_1 + \alpha_2$ is not a root.

Summary.

- $[H_i, E_\beta] = \langle \alpha_i, \beta \rangle E_\beta$. where $\langle \alpha_i, \alpha_j \rangle = (H_i, H_j)$.

- Semisimple Lie algebras are encoded in the Dynkin diagrams $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$