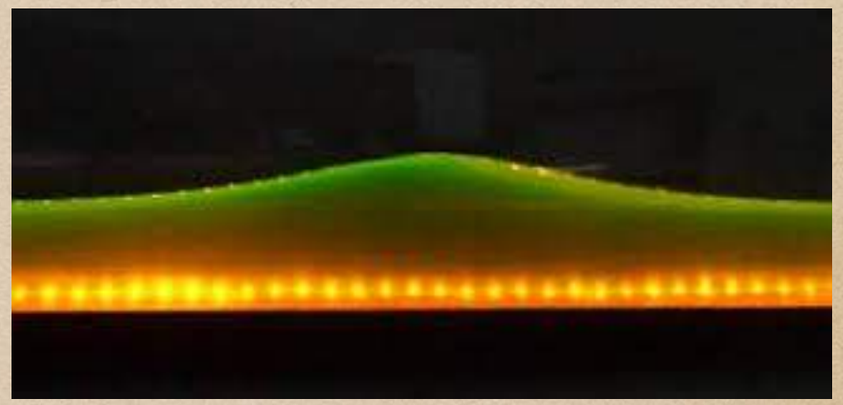
The Mathematics of Waves

from Catastrophe to Solitons





Dr Paul P. Cook

Department of Mathematics

King's College London





Aims:

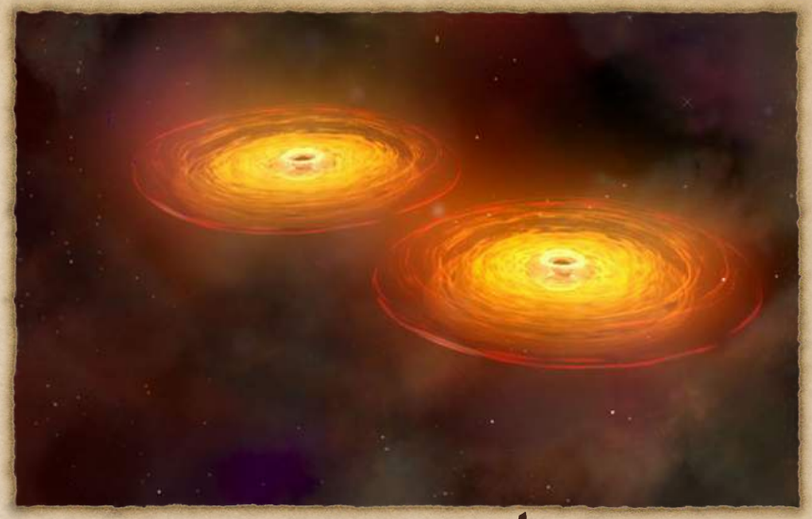
1. Understand why waves spread or break. (CATASTROPHE!)

2. Discover that some waves remain perfectly stable. (STABILITY!)

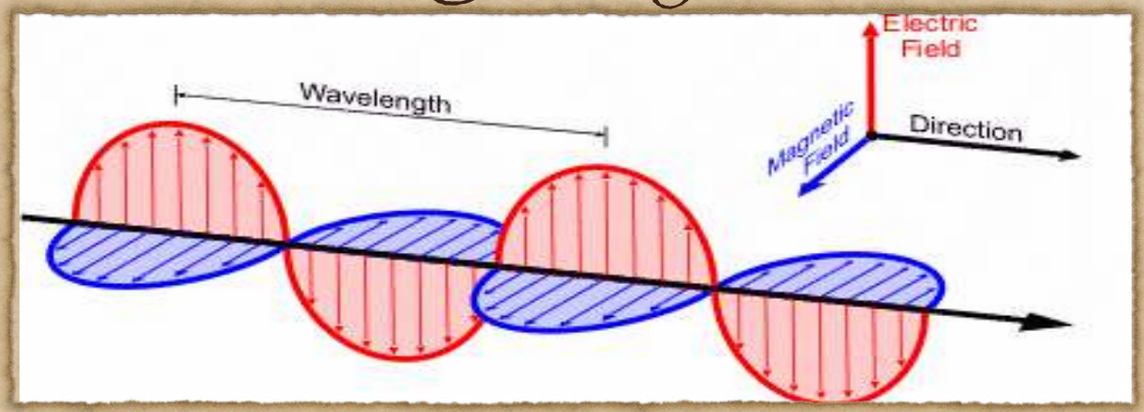
3. Study the equation that produces stable waves in shallow water: The KdV equation. (MATHS)



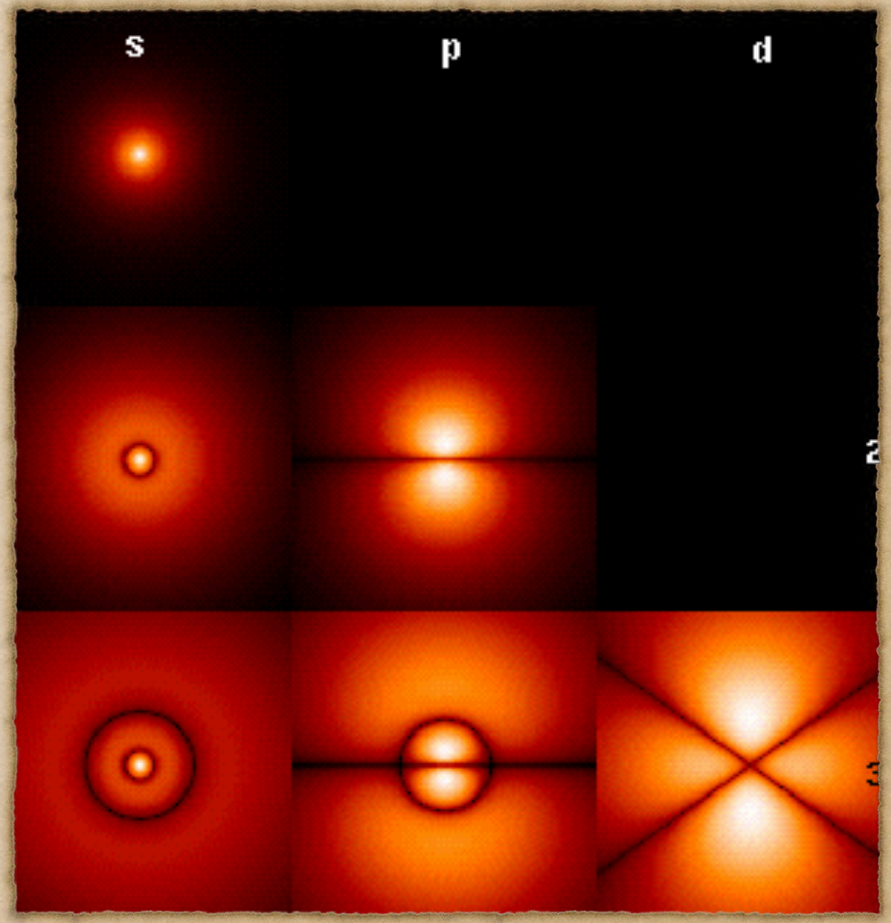
Water (gravity) waves



Gravitational waves



Electromagnetic waves



Probability waves

Two Million Dollar Motivations

Yang-Mills and the Mass Gap

‘Can one construct 4D quantum Yang–Mills theory rigorously, and prove it has a positive mass gap?’

(Fields can form stable structures (e.g. flux tubes between quarks), closely related to soliton behaviour.)

Navier-Stokes

‘Do smooth three-dimensional fluid flows stay smooth for all time, or can singularities form?’

Wave Equations

A one-dimensional wave equation

$$u_{tt} = c^2 u_{xx}$$

Notation

$$u_t = \frac{du}{dt}, \quad u_{tt} = \frac{d^2u}{dt^2} \cdots$$

$$u_x = \frac{du}{dx}, \quad u_{xx} = \frac{d^2u}{dx^2} \cdots$$

where u(x,t) is the amplitude of a wave.

Wave Equations

$$u_{tt} = c^2 u_{xx}$$

- Solution:

$$u(x,t) = f(x - ct) + g(x + ct)$$

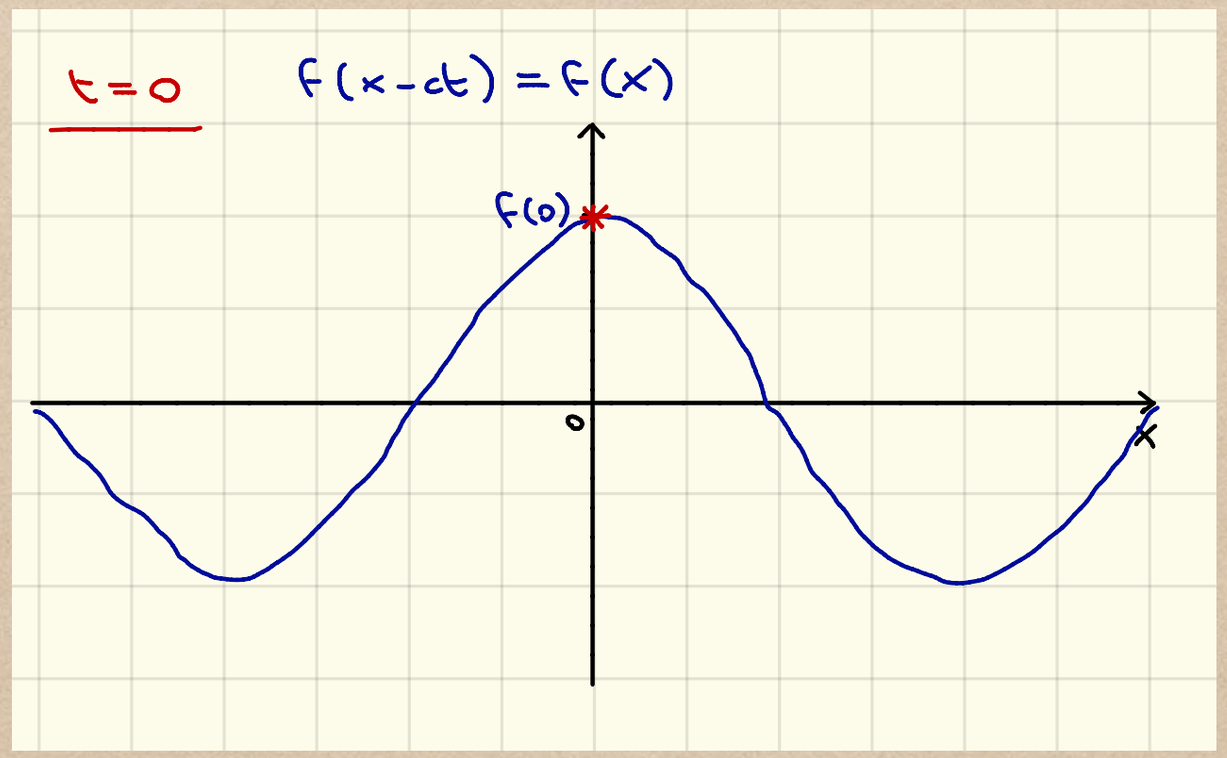
where f and g are any functions.

- This is an infinite set of solutions!

Moving Wave Profiles

Consider $u(x,t) = f(x - ct)$

It is a wave whose shape is unchanged and moves to the right at speed c



The point at x=0 initially (t=0) corresponds to x-ct=0, later we have x=ct to keep the argument zero.

The Linear Wave Equation

- The wave equation is linear in u: If v is a solution and w is a solution the v+w are solutions too.
- u=f(x-ct) moves to the right with speed c
- u=g(x+ct) moves to the left with speed c
- f(x-ct) is a solution to a simpler equation:

$$u_t + cu_x = 0$$

- N.B.

$$\left(\frac{d^2}{dt^2} - c^2\frac{d^2}{dx^2}\right)u = \left(\frac{d}{dt} - c\frac{d}{dx}\right)\left(\frac{d}{dt} + c\frac{d}{dx}\right)u = 0$$

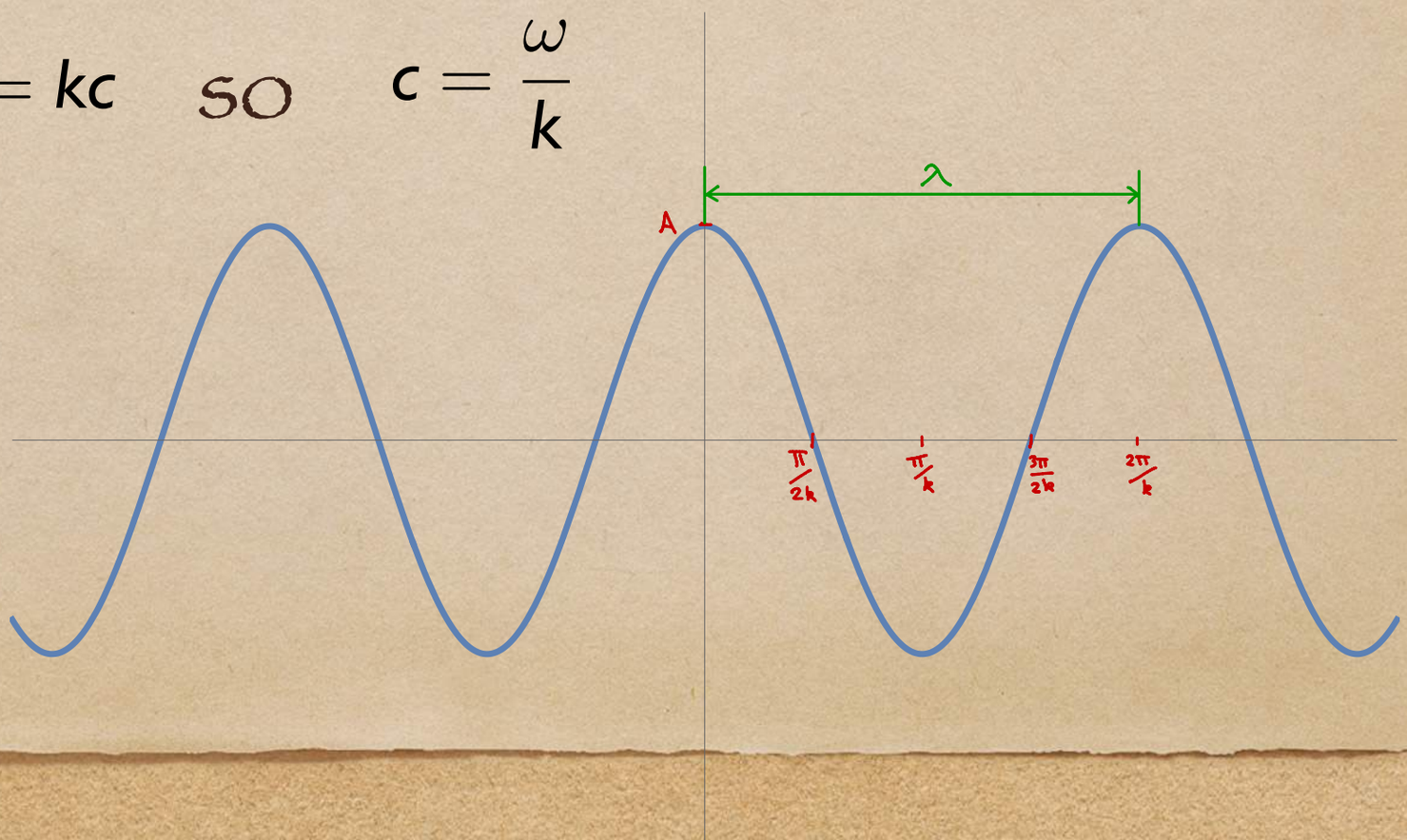
$u_t - cu_x = 0$ $\qquad$ $u_t + cu_x = 0$

Harmonic Travelling Waves

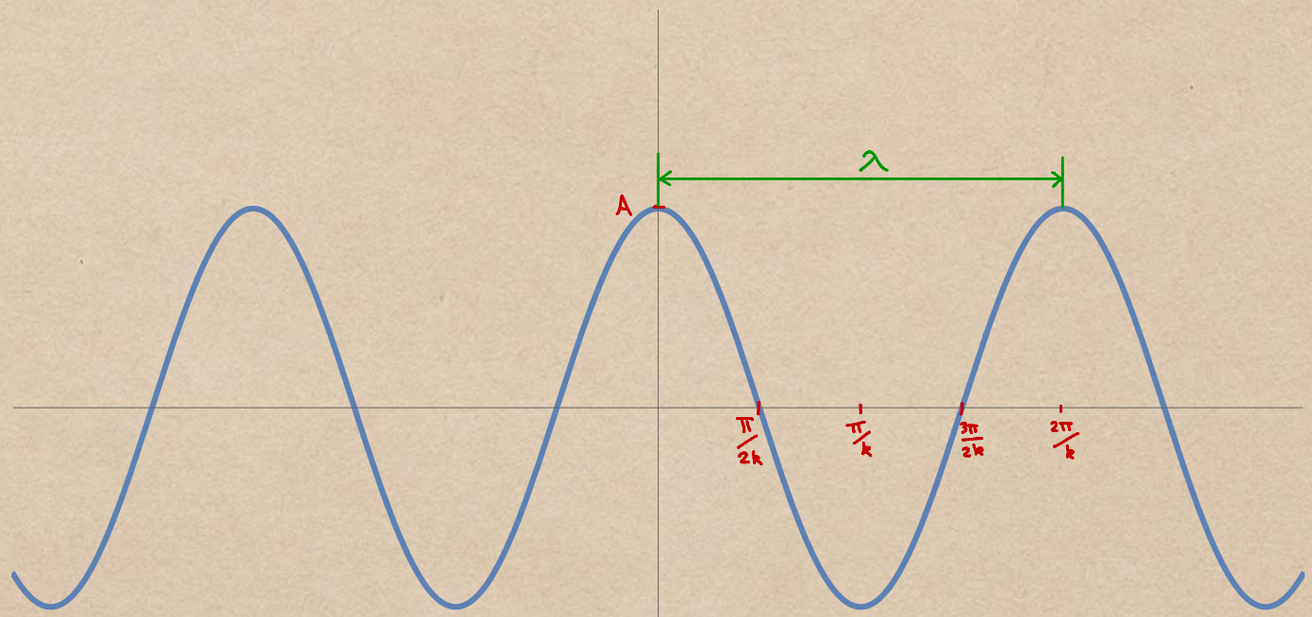
- Harmonic travelling waves form the building blocks of many travelling wave solutions.
- are sine or cosine wave trains e.g.

$$u = A\cos(k(x - ct)) = A\cos(kx - \omega t)$$

- Where $\omega = kc$ so $c = \frac{\omega}{k}$
- At t=0



Travelling Waves

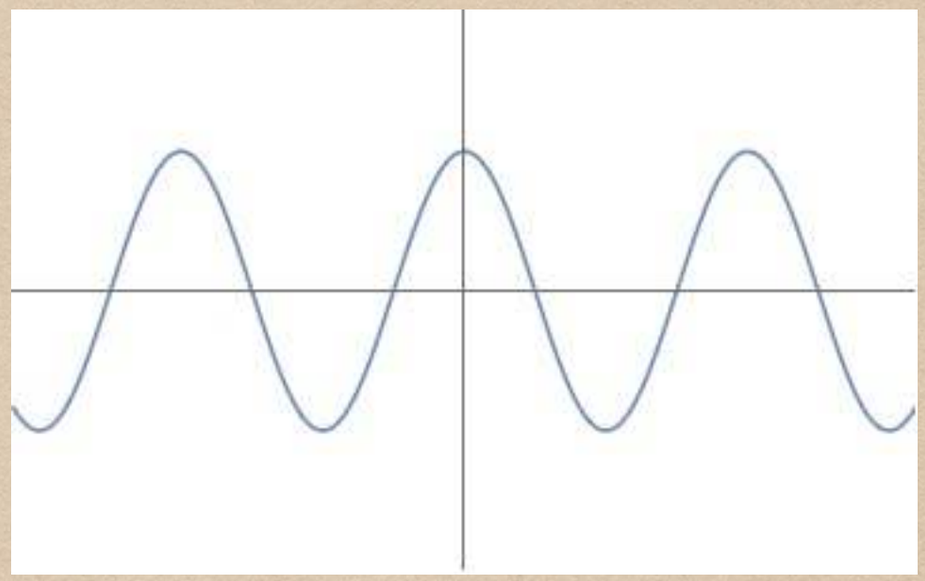


$$u = A\cos(kx - \omega t) \qquad c = \frac{\omega}{k} \qquad \lambda = \frac{2\pi}{k}$$

A wave travels one wavelength in $\frac{\lambda}{c} = \frac{2\pi}{\omega}$

Travelling Waves Summary

$$u = A\cos(kx - \omega t)$$



- This is a travelling wave solution to $u_t + cu_x = 0$
- Its wave speed is constant $\frac{\omega}{k} = c$
- Solutions can be added together to find new solutions: it is linear.

Addition of Waves

- A travelling wave is the real part of

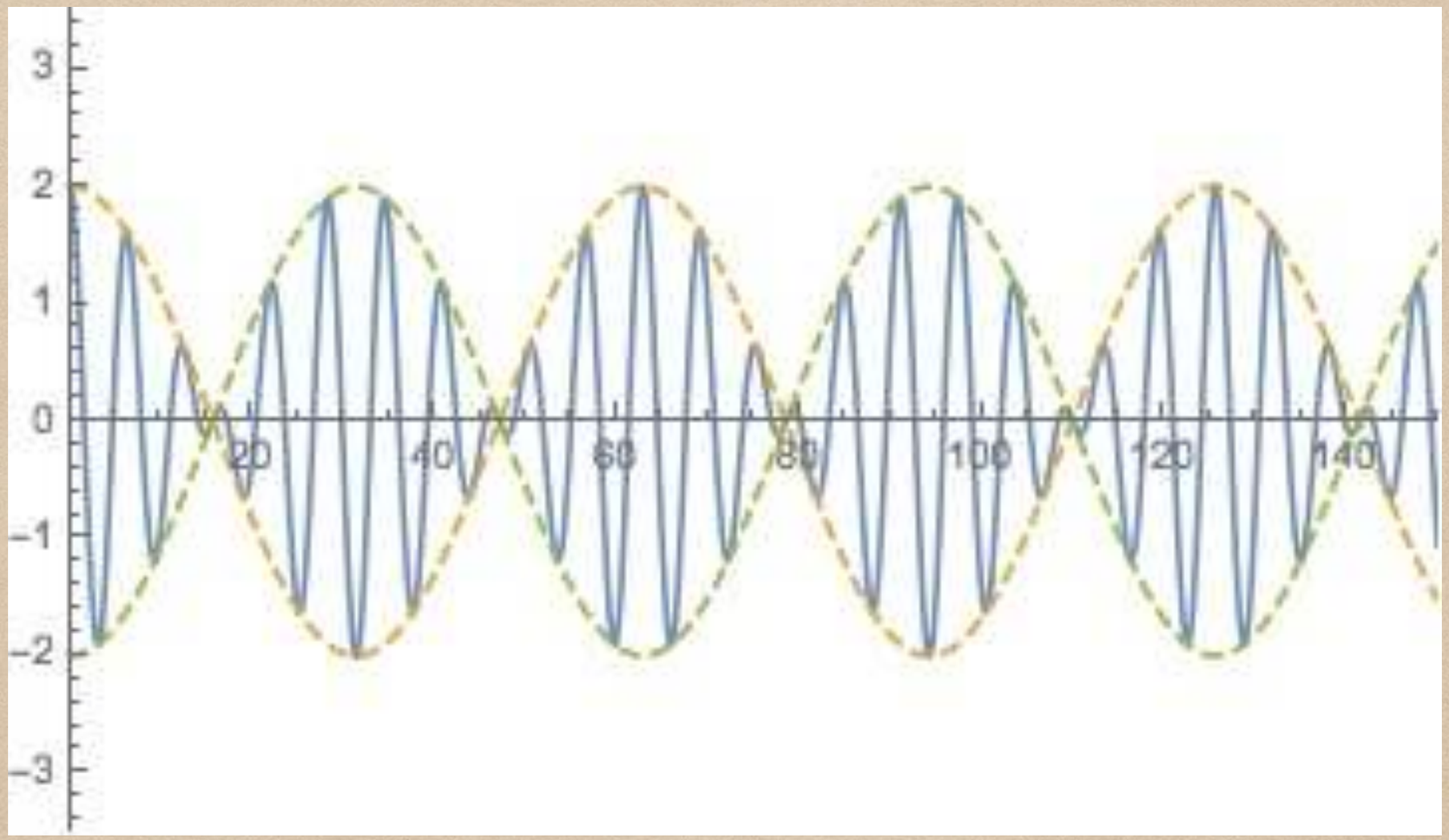
$$u = Ae^{i(kx-\omega t)}$$

$$= A\cos(kx - \omega t) + iA\sin(kx - \omega t)$$

- The angular frequency w=ck is a function of k.
- We could add many travelling waves together to find another wave

$$u = \sum_k A(k)e^{i(kx-\omega t)}$$

Addition of Waves: Beats

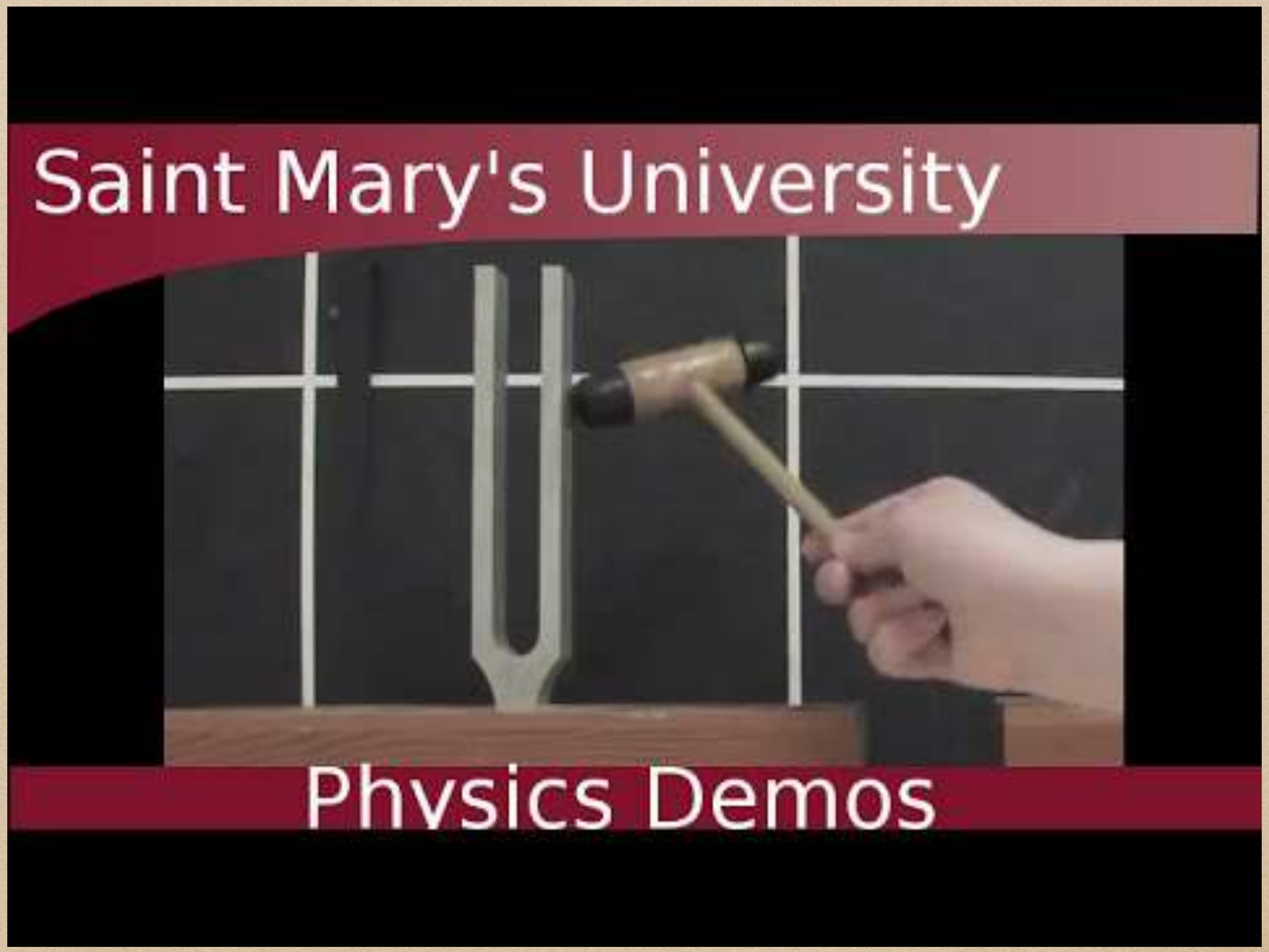


$$u = \cos\Big((k-\delta k)x-(\omega-\delta\omega)t\Big)+\cos\Big((k+\delta k)x-(\omega+\delta\omega)t\Big)$$

$k = 1 \quad \delta k = 0.1$

$\omega = 1 \quad \delta\omega = 0.2$

Addition of Waves: Beats



Courtesy of St Mary's University, California

Dissipation

Let's modify the wave equation from

$$u_t + cu_x = 0$$

to:

$$u_t + cu_x + du_{xx} = 0$$

There is a travelling wave solution $u = Ae^{i(kx-\omega t)}$ if

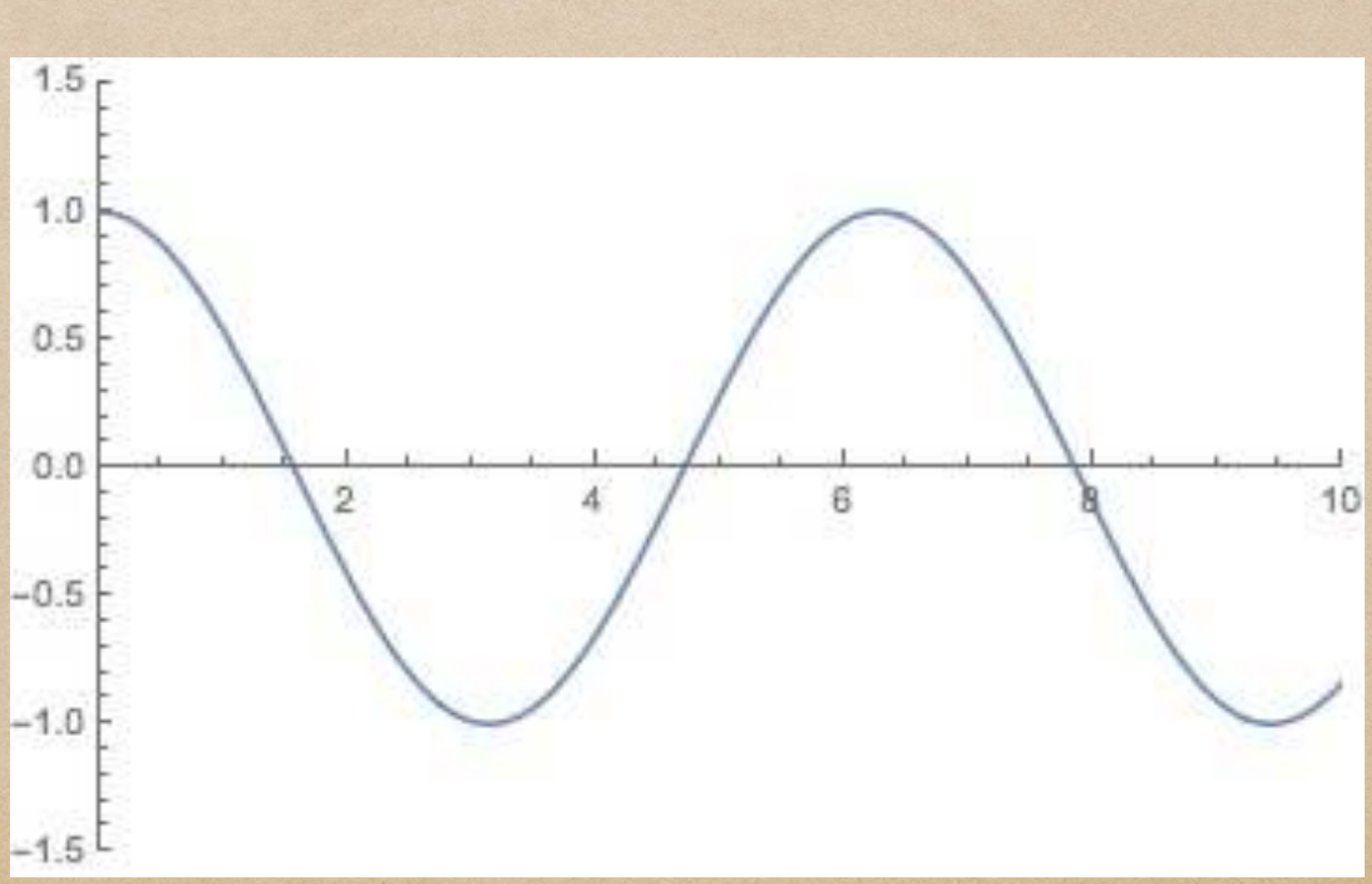
$$\omega = ck + idk^2$$

giving

$$u = Ae^{ik(x-ct)}e^{dk^2 t}$$

Dissipation

If d>0 the wave amplitude grows, while if d<0 the wave dissipates:



Dispersion

Let's consider adding a term with three x derivatives:

$$u_t + cu_x + du_{xxx} = 0$$

This has a travelling wave solution $u = Ae^{i(kx-\omega t)}$

with dispersion relation

$$\omega = ck - dk^3$$

and wave speed which depends on wavelength:

$$\frac{\omega}{k} = c - dk^2 = c - d\frac{4\pi^2}{\lambda^2}$$

Dispersion

$$\frac{\omega}{k} = c - dk^2 = c - d\frac{4\pi^2}{\lambda^2}$$

Now if

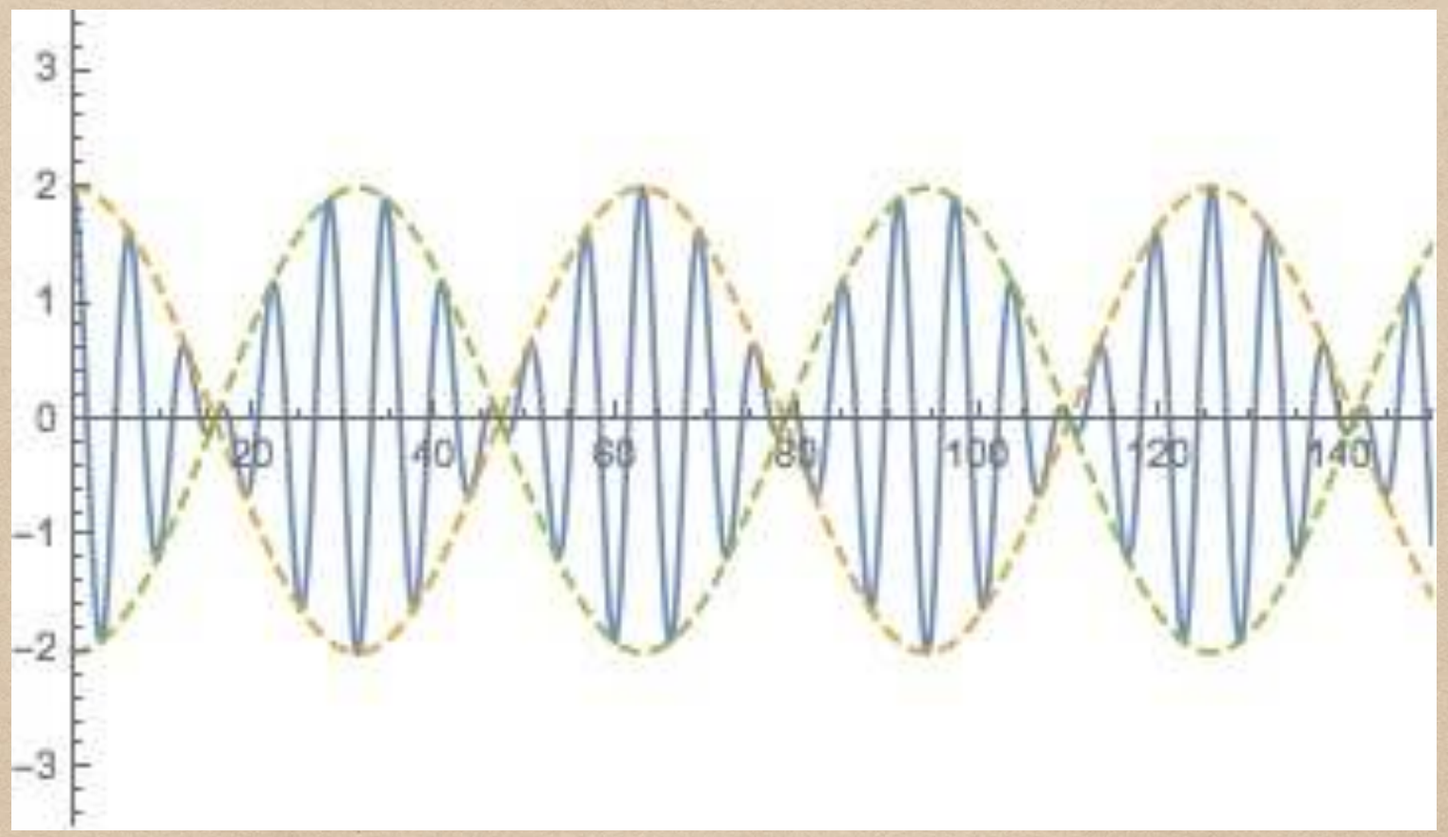
$$\lambda^2 = \frac{4\pi^2 d}{c}$$

the wave has zero wave speed!

A wave travels to the right if $\lambda^2 > \frac{4\pi^2 d}{c}$

or it travels to the left if $\lambda^2 < \frac{4\pi^2 d}{c}$

Dispersion



Wave 1 had wave speed 8/9 but with dispersion it now has speed 71/900

Wave 2 had wave speed 12/11 but with dispersion it now has speed -131/1100

Dispersion

Gravity waves in deep water: $\omega = \sqrt{gk}$

$$\frac{\omega}{k} = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}}$$

$$\frac{d\omega}{dk} = \frac{1}{2}\sqrt{\frac{g}{k}}$$

Long wavelengths travel faster.

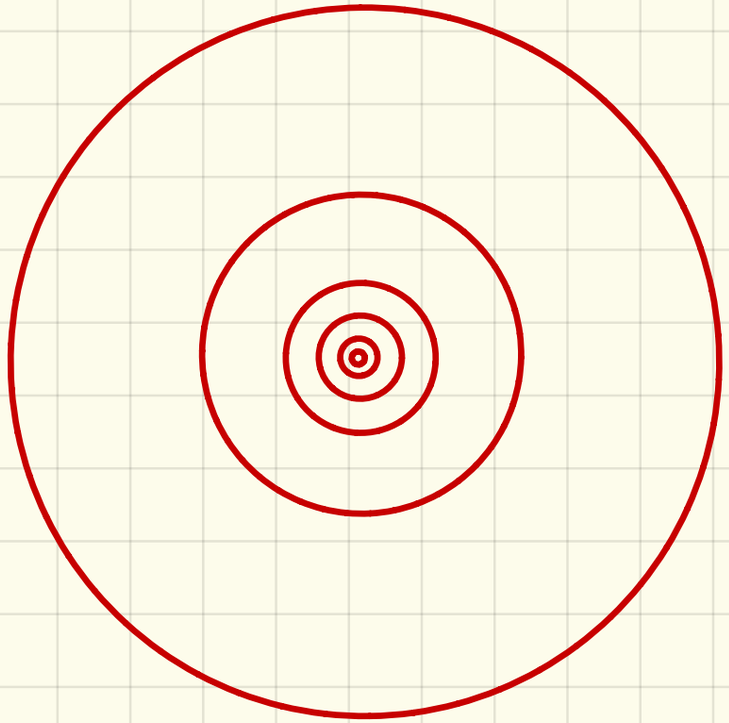
Capillary waves: $\omega = \sqrt{\frac{T|k|^3}{\rho}}$

$$\frac{\omega}{k} = \sqrt{\frac{T|k|}{\rho}} = \sqrt{\frac{2T\pi}{\lambda\rho}}$$

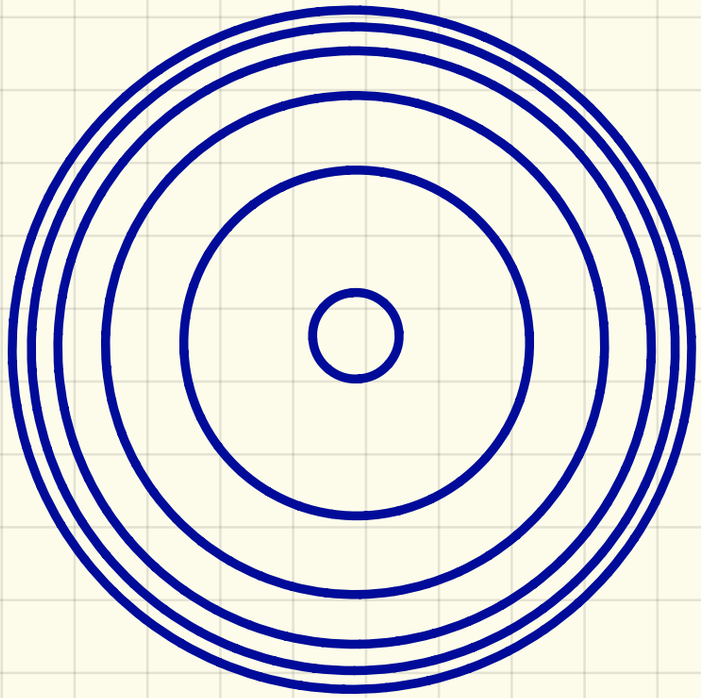
$$\frac{d\omega}{dk} = \frac{3}{2}\sqrt{\frac{T|k|}{\rho}}$$

Short wavelengths travel faster.

Dispersion



Gravity Wave (e.g. a stone dropped in water)



Capillary Wave (e.g. a raindrop falling on water)

Non-Linear Terms

Could we make wave-speed a function of amplitude?

Consider:

A non-linear term in u.

$$u_t + (c+u)u_x = 0$$

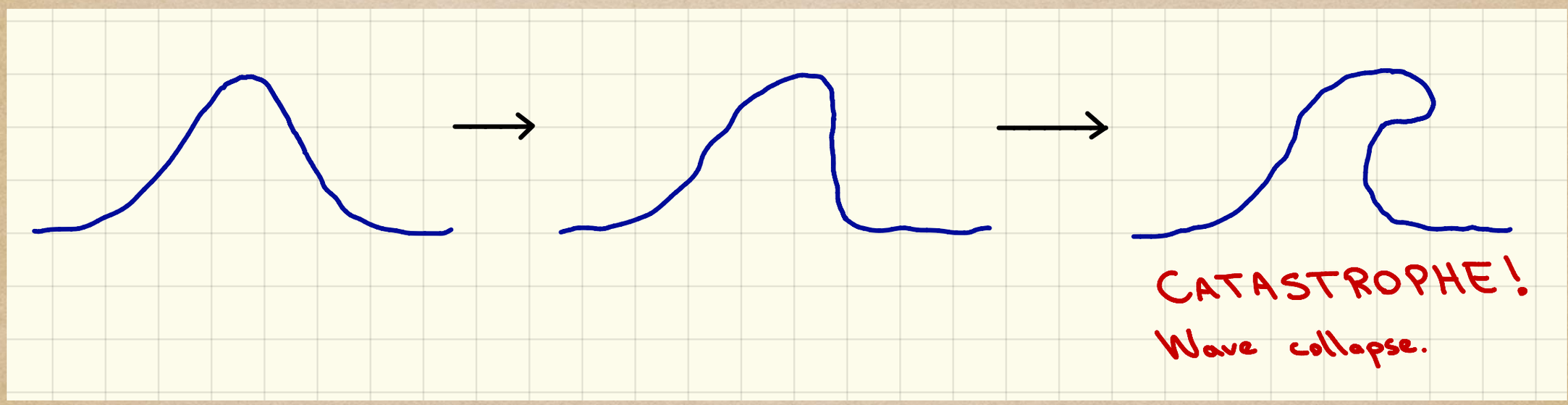
Points carrying value u now move at c+u - they depend on the amplitude of the wave, u.

$$\frac{du(x(t),t)}{dt} = u_t + u_x\frac{dx}{dt} = 0 \implies \frac{dx}{dt} = c+u$$

on the characteristic curve x(t).

Non-linear terms

Taller amplitudes have a faster wave speed:



We can no longer add solutions to find a new one (due to non-linearity).



The Great Wave off Kanagawa by Hokusai

Dispersive, Non-linear Wave Equations

$$u_t + (c + u)u_x + u_{xxx} = 0$$

Non-linear term (wave-speed depends on amplitude)

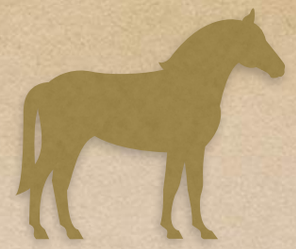
Dispersion term (wave-speed depends on wavelength)

Can we find solutions for which dispersion and amplitude dependence cancel out?

In fact an equation having this form describes shallow water waves.



Solitons



"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation."

John Scott Russell, 1834

Solitons

$$u_t + 6uu_x + u_{xxx} = 0$$

Non-linear term

Dispersion term

This is the Korteweg-de Vries equation (or KdV equation) which governs the motion of shallow water.

[Korteweg-de Vries 1895]

It has a travelling wave solution:

$$u = \frac{c}{2\cosh^2(\frac{1}{2}\sqrt{c}(x - ct))}$$

Look for a travelling wave solution $u = f(x - ct) = f(\xi)$ to

$u_t + 6uu_x + u_{xxx} = 0.$

Substitution gives:

$$-cf' + 6ff' + f''' = (-cf + 3f^2 + f'')' = 0$$

Integrating with the boundary condition that f, f' vanish at infinity:

$$-cf + 3f^2 + f'' = 0 \implies -cff' + 3f^2 f' + f'' f' = \frac{d}{d\xi}(-\frac{c}{2}f^2 + f^3 + \frac{1}{2}(f')^2) = 0$$

and boundary conditions imply $f' = \sqrt{cf^2 - 2f^3}$. We have

$$\int \frac{df}{f\sqrt{c - 2f}} = \int d\xi$$

Substituting $f = \frac{c}{2}\text{sech}^2\alpha$ leads to $\alpha = -\frac{\sqrt{c}}{2}\xi$ and so

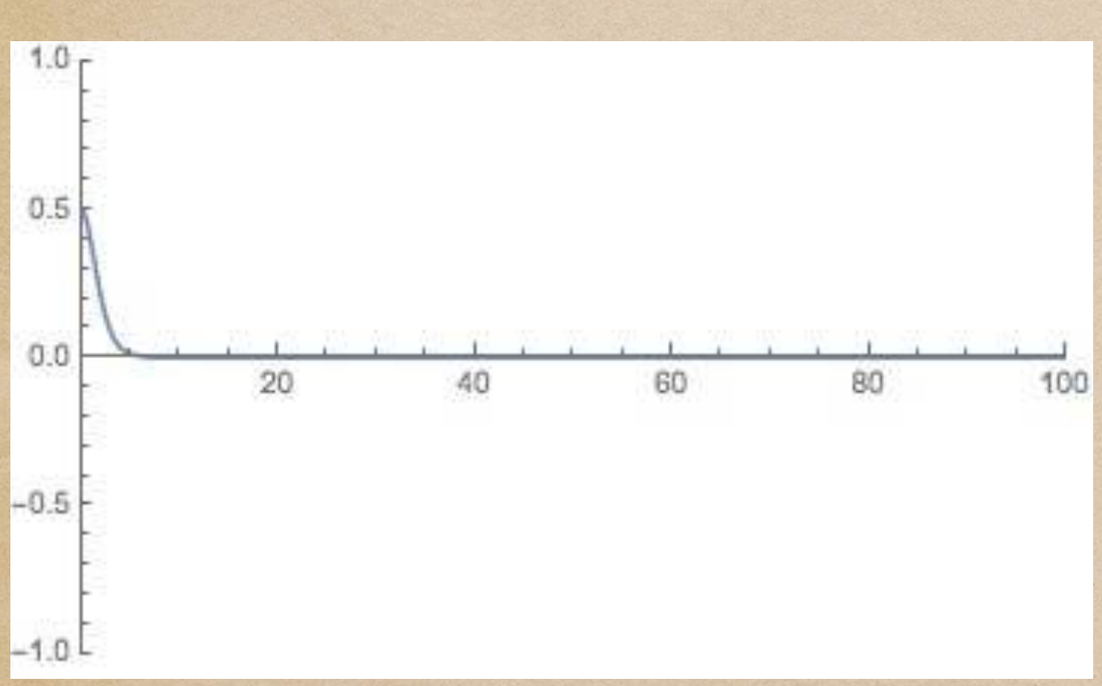
$$u = \frac{c}{2}\text{sech}^2(\frac{\sqrt{c}}{2}(x - ct))$$

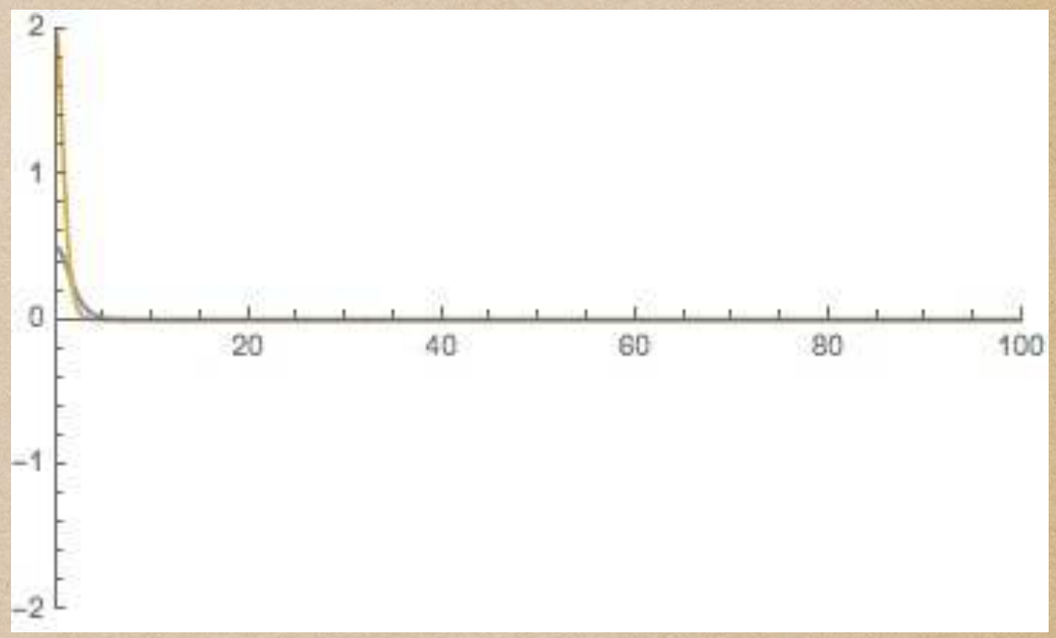
Solitons

$$u_t + 6uu_x + u_{xxx} = 0$$

$$u = \frac{c}{2\cosh^2(\frac{1}{2}\sqrt{c}(x - ct))}$$

The height of the soliton is proportional to its speed.



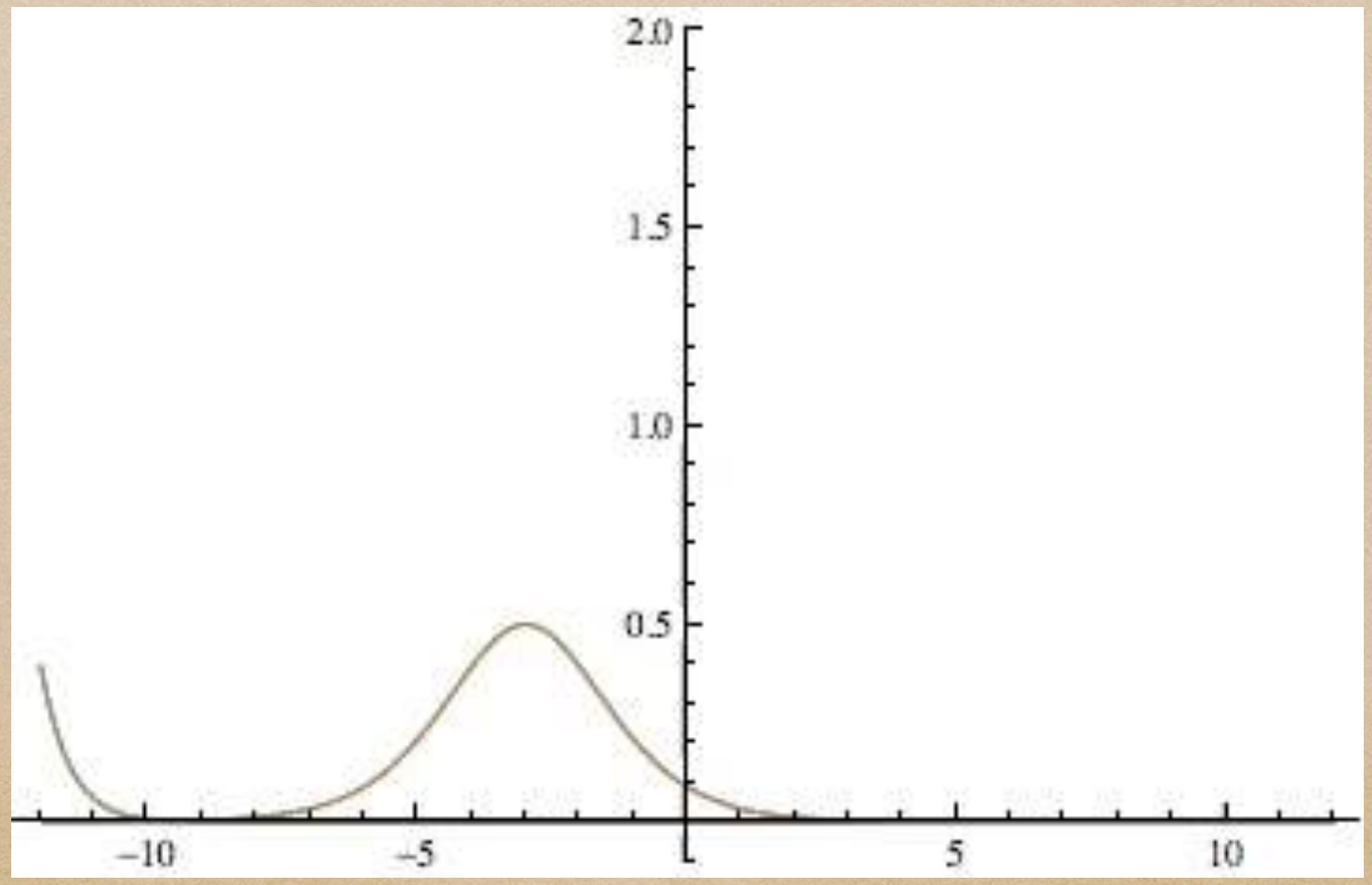


Solitons

Although the KdV equation is non-linear there are techniques to solve it and find many-soliton solutions.

Incredibly when they collide they retain their structure!

[Soliton interactions: Zabusky & Kruskal 1965]





Credit: Christophe Finot, Laboratoire Interdisciplinaire CARNOT de Bourgogne, Équipe Solitons, Laser et Communications optiques, With soundtrack by Ennio Morricone (The Ecstasy of Gold)

Integrable Equations

Why are solitons so stable?

The KdV equation is an example of an integrable p.d.e., these are equations which have an infinite number of conserved quantities:

$$Q_1 = \int u dx, \quad Q_2 = \int u^2 dx, \quad Q_3 = \int (u_x^2 - 2u^3) dx \ \cdots$$

These constraints restrict how the wave can evolve.

Result: the wave profile cannot spread or change shape.

In finite-dimensional systems, integrability means having as many conserved quantities as degrees of freedom.

Integrable Equations

Examples of integrable systems include:

- The KdV Equation (shallow water waves)

- The nonlinear Schrödinger Equation (optics, Bose-Einstein condensates)

- Sine-Gordon Equation (kinks and topological solitons)

- Toda lattice (nonlinear interacting particles)

- Heisenberg spin chains (quantum integrable model)

Inverse Scattering

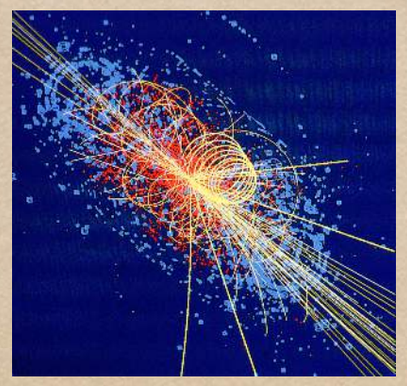
Gardner, Greene, Kruskal and Miura discovered (1967) that the KdV equation can be solved using the inverse scattering equation.

Idea: Instead of solving the nonlinear KdV equation directly, we turn it into a linear problem (a Schrödinger equation) and solve that instead.

$$-\psi_{xx} + u(x,t)\psi = \lambda\psi \qquad cf. \qquad \hat{H}\psi = \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\psi = E\psi$$

The wave profile acts like a potential in a Schrödinger equation.

By following how waves scatter off the potential u, we can construct u.



[N.B. to obtain the equation above use the Miura transform $u = v^2 + v_x$, together with $v = \frac{\psi_x}{\psi}$ to obtain the $\lambda = 0$ case.]

Lax Pairs

A Lax pair is a way of rewriting the equation so that the eigenvalues of an operator don't change in time.

$-\psi_{xx} + u(x,t)\psi = \lambda\psi$ may be recast as an operator equation:

$$L\psi = \lambda\psi \qquad \text{where} \qquad \mathrm{L(x,t)} = -\frac{\partial^2}{\partial \mathrm{x}^2} + \mathrm{u(x,t)}$$

Together with the compatibility condition

$$[L, B] = L_t$$

Where B=B(x) is the operator satisfying

$$B\psi = -\frac{\partial \psi}{\partial t}$$

Nonlinear evolution is replaced by an operator equation.

Systems constructed this way have constant eigenvalues, which makes them solvable.

Summary Comments

- Wave equations are mathematically interesting and the applications are wide.
- Certain equations are more interesting than others (integrable equations).
- The mathematics involved is predominantly:
    - vectors (scalar and cross-product),
    - calculus (derivatives, integrals), and
    - vector calculus (directional derivatives, line integrals).
- It is an active area of mathematical research.

Thank-you.

Lax Pairs

$$-\psi_{xx} + u(x,t)\psi = \lambda\psi$$

May be recast as an operator equation:

$$L\psi = \lambda\psi \qquad \text{where} \qquad \mathrm{L(x,t)} = -\frac{\partial^2}{\partial \mathrm{x}^2} + \mathrm{u(x,t)}$$

Time evolution:

Write $U(\tau) = e^{\tau\frac{d}{dt}}$ then

$$U(\tau)L(x,t) = (1 + \tau\frac{d}{dt} + \frac{\tau^2}{2!}\frac{d}{dt} + \ldots)L(x,t) = L(x,t+\tau)$$

$$L(x,t+\tau)\psi(x,t+\tau) = \lambda\psi(x,t+\tau)$$

$$\implies L(x,t+\tau)U\psi(x,t) = \lambda U\psi(x,t)$$

Or $U^{-1}L(x,t+\tau)U = L(x,t)$ equivalently $\mathrm{UL(x,t)U^{-1} = L(x,t+\tau)}$

Lax Pairs

Writing $L(0) = U^{-1}L(t)U$ (now with $U = U(t)$) then taking the t derivative and rearranging gives:

$$L_t = U_t U^{-1} L - L U_t U^{-1} = -BL + LB = [L, B]$$

Where $B = -U_t U^{-1}$, so the Lax pairs are the operators

L and B satisfying the compatibility condition $[L, B] = L_t$

Such a system guarantees that the eigenvalue problem $L\psi = \lambda\psi$ has constant eigenvalues.

Lax Pairs

$$L_t\psi + L\psi_t = \lambda_t\psi + \lambda\psi_t \implies (L_t - \lambda_t)\psi = (\lambda - L)\psi_t$$

Using the compatibility condition

$$(LB - BL - \lambda_t)\psi = (\lambda - L)\psi \implies (L - \lambda)(\psi_t + B\psi) = \lambda_t\psi$$

An inner product gives

$$\lambda_t |\psi|^2 = \langle (L - \lambda)(\psi_t + B\psi) \,|\, \psi \rangle = \langle \psi_t + B\psi \,|\, (L - \lambda)\psi \rangle = 0$$

Where we took L to self-adjoint.

Lax Pairs

The question can now be asked: what is B(x) which ensures that $B\psi = -\frac{\partial \psi}{\partial t}$ given that L is a second-order differential operator.

B is first order: $B_1 = C_1 \frac{\partial}{\partial x}$ is the generator of x translations

B is second order: $B_2 = C_2 \frac{\partial^2}{\partial x^2} + D_2 u \implies [B_2, L] = 0.$

B is third order: $B_3 = C_1 \frac{\partial^2}{\partial x^2} + D_2 u \frac{\partial}{\partial x} + E_3 u_x$ solving the compatibility equation gives the KdV equation from $B\psi = -\psi_t$.