Reference Frames - from the Coriolis Effect to Time Dilation.

(Or "Galileo, Newton, Einstein.")

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What is a Reference Frame?

Definition

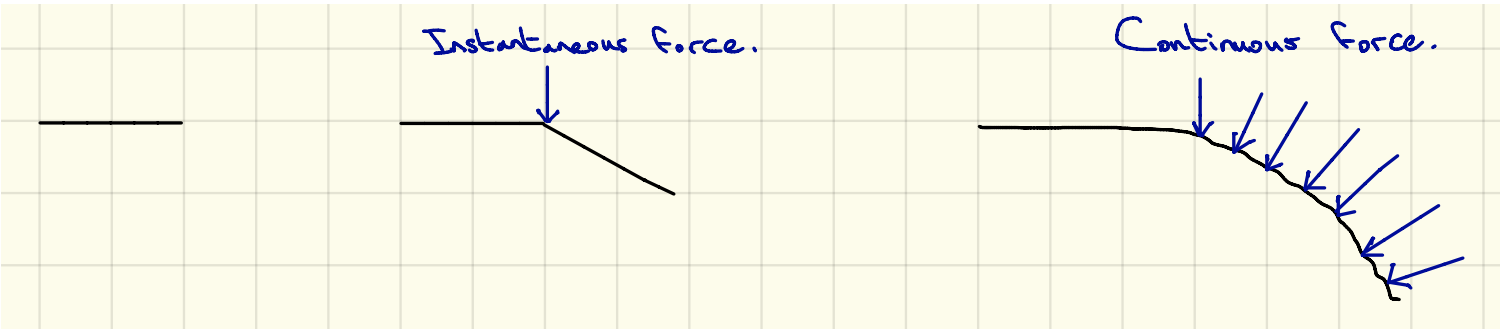
A reference frame, or a frame of reference, is a coordinate system.

Comments

- *A coordinate system gives a set of numbers (coordinates) to each point in a geometric space, e.g. Cartesian coordinates or polar coordinates*
- *Language: one may loosely refer to the laboratory as the reference frame or perhaps a moving vehicle, what is implied is that the choice of coordinates is stationary with respect to the setting.*

Circular Motion

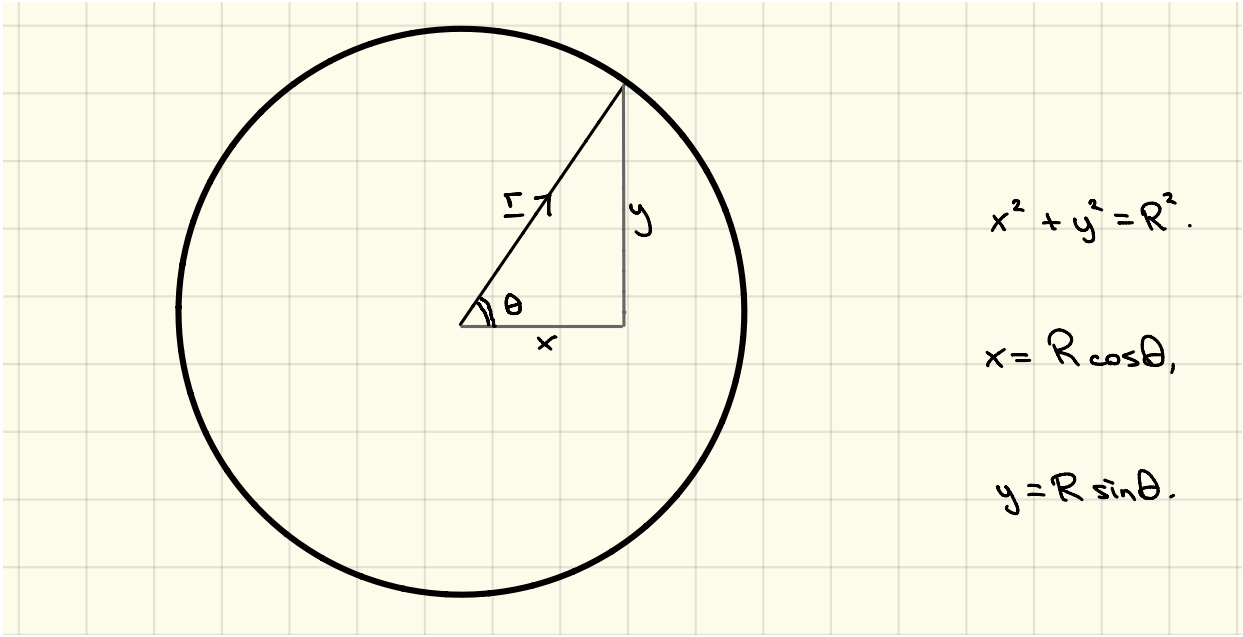
What force is applied to give a circular motion?



But how do we show this mathematically?

Circular Motion

Assume a particle of mass m moves on a circle of radius R:



The radius R is constant so,

$$\dot{x} = -(R\sin\theta)\dot{\theta}, \qquad \dot{y} = (R\cos\theta)\dot{\theta}$$
$$\ddot{x} = -(R\cos\theta)\dot{\theta}^2, \qquad \ddot{y} = -(R\sin\theta)\dot{\theta}^2$$

Circular Motion

Then, by Newton's second law,

$$\mathbf{F} = m\mathbf{a} = m\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = m\begin{pmatrix} -(R\cos\theta)\dot{\theta}^2 \\ -(R\sin\theta)\dot{\theta}^2 \end{pmatrix} = -mR\dot{\theta}^2\hat{\mathbf{r}}$$

where $\hat{\mathbf{r}}$ is the unit radial vector.

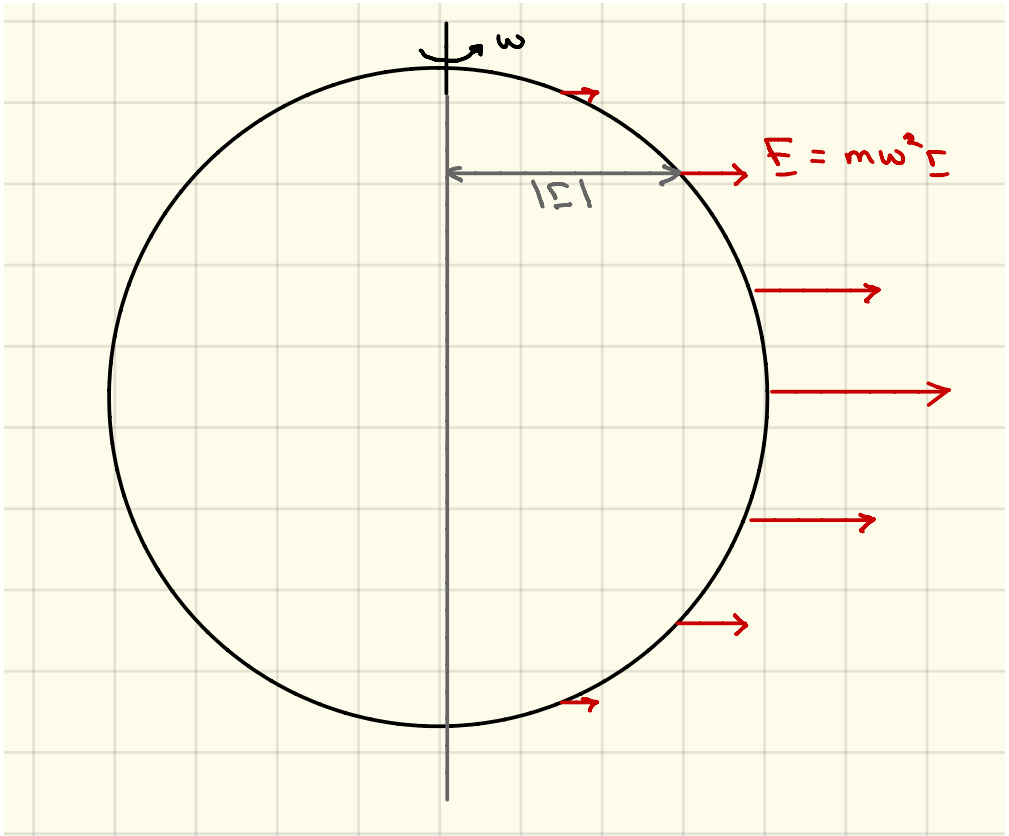
Comments

- *There is an acceleration!*
- *So if you drove in a car around the Earth at a constant speed of 30mph you would be accelerating.*
- *Alternatively if you stand still and the Earth rotates, there is a force acting on you (the centripetal force).*
- *If we had worked in a rotating frame there would have been no acceleration.*

Circular Motion

By Newton's third law we feel an outward reaction called the centrifugal force due to the earth rotating at $\dot{\theta} = \omega$:

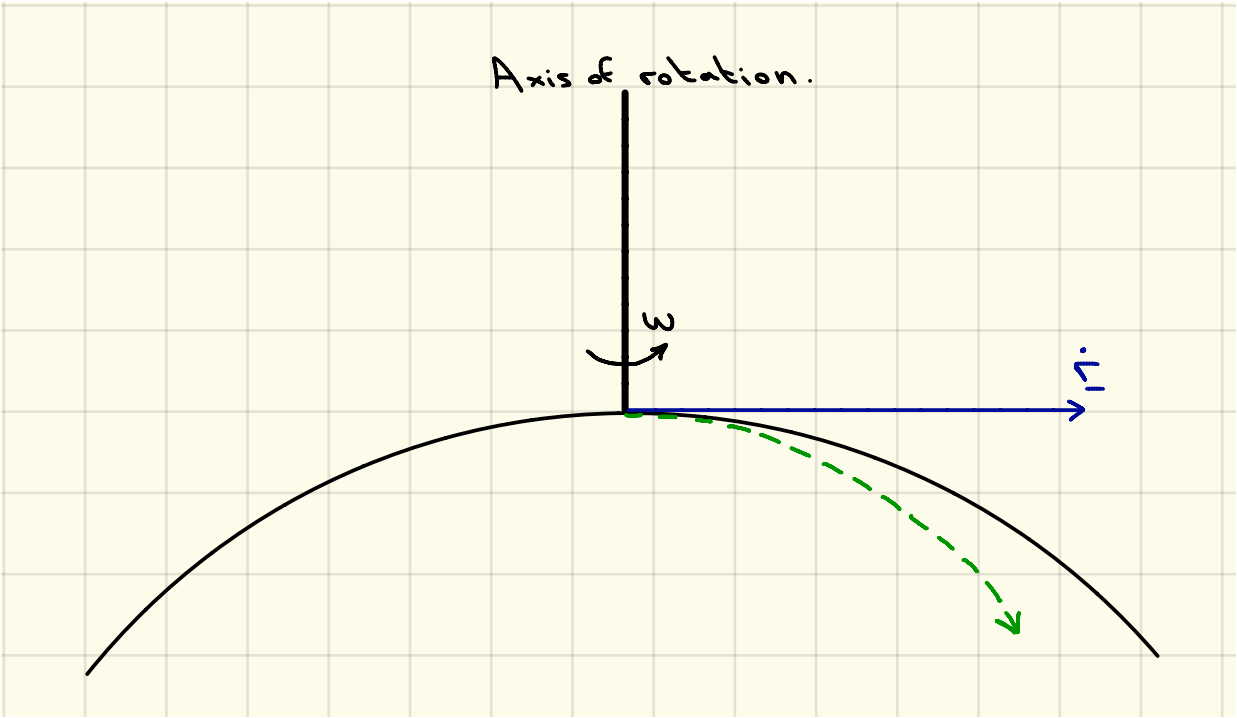
$$\mathbf{F} = m\omega^2\mathbf{r}$$



You are 0.3% lighter at the equator.

The Coriolis Effect

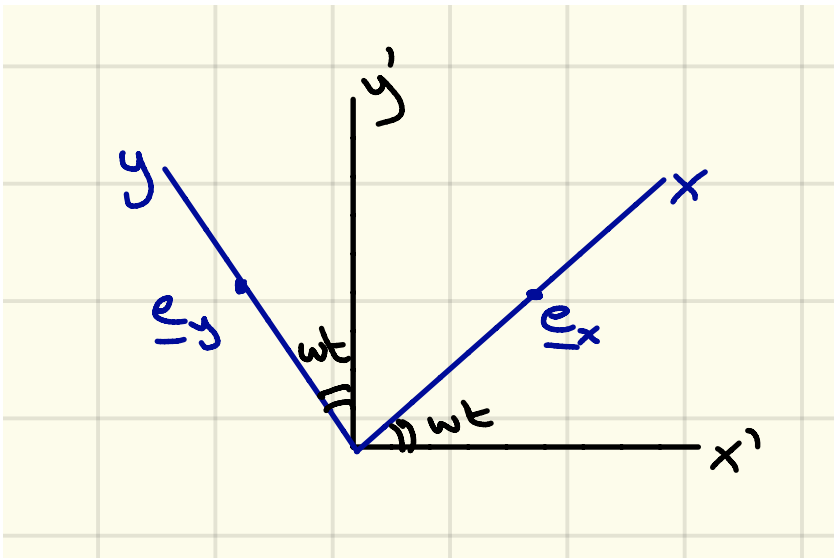
The Coriolis effect is responsible for the deflection of particles as judged from the rotating frame of reference of the Earth.



A particle moving in a straight line through space of speed $\dot{\mathbf{r}}$ has a curved path in a rotating frame.

The Coriolis Effect

We can mathematically find the apparent force causing the curved path by changing to a rotating frame of coordinates:



$$\mathbf{e_x} = \cos(\omega t)\mathbf{e'_x} + \sin(\omega t)\mathbf{e'_y}$$

$$\mathbf{e_y} = -\sin(\omega t)\mathbf{e'_x} + \cos(\omega t)\mathbf{e'_y}$$

$$\mathbf{r} = x\mathbf{e_x} + y\mathbf{e_y} = \begin{pmatrix} x\cos(\omega t) - y\sin(\omega t) \\ x\sin(\omega t) + y\cos(\omega t) \end{pmatrix}$$

The Coriolis Effect

$$\begin{aligned}\mathbf{F} &= m\ddot{\mathbf{r}} \\ &= m\omega \begin{pmatrix} -2\dot{x}\sin(\omega t) - 2\dot{y}\cos(\omega t) + \omega y \sin(\omega t) - \omega x \cos(\omega t) \\ 2\dot{x}\cos(\omega t) - 2\dot{y}\sin(\omega t) - \omega x \sin(\omega t) - \omega y \cos(\omega t) \end{pmatrix} \\ &= 2m\omega \begin{pmatrix} -\dot{y}' \\ \dot{x}' \end{pmatrix} + m\omega^2 \begin{pmatrix} x' \\ y' \end{pmatrix}\end{aligned}$$

Comments

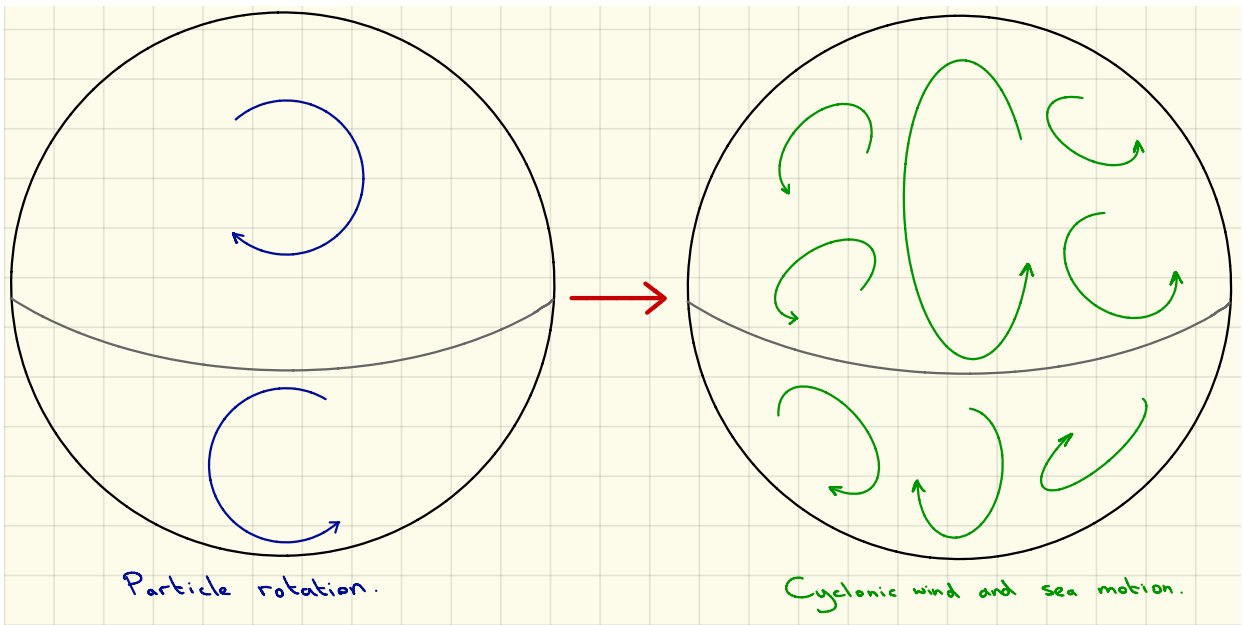
- *If* $\omega = 0$ *both of these forces vanish.*
- *In the second term we recognise the centrifugal force* $m\omega^2\mathbf{r}'$.
- *The first term is called the Coriolis force*

$$\mathbf{F}_{Coriolis} = 2m\omega \begin{pmatrix} -\dot{y}' \\ \dot{x}' \end{pmatrix}$$

The Coriolis Effect

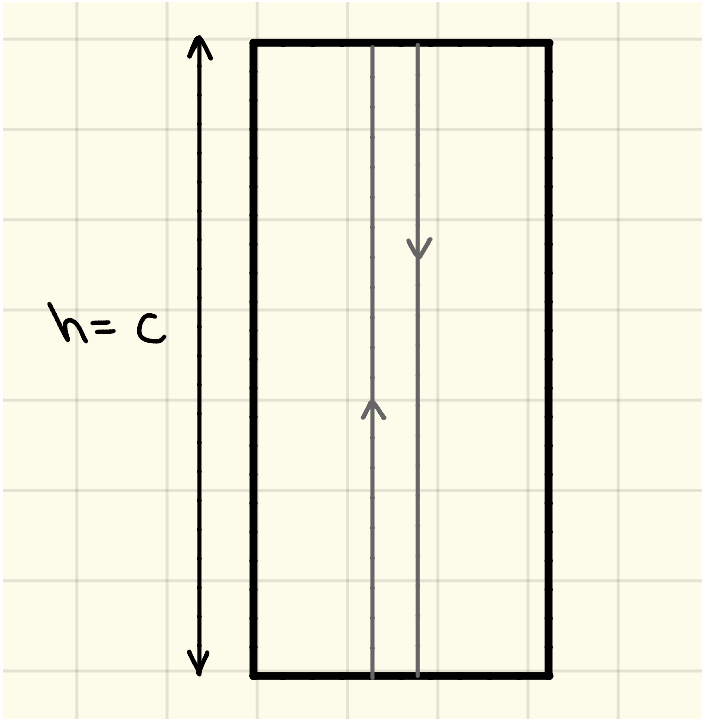
$$\mathbf{F}_{\text{Coriolis}} = 2m\omega \begin{pmatrix} -\dot{y}' \\ \dot{x}' \end{pmatrix}$$

The Coriolis force is perpendicular to the velocity vector of the particle $\dot{\mathbf{r}}' = \begin{pmatrix} \dot{x}' \\ \dot{y}' \end{pmatrix}$. Hence it drives moving particles to move in circular paths.



Time Dilation

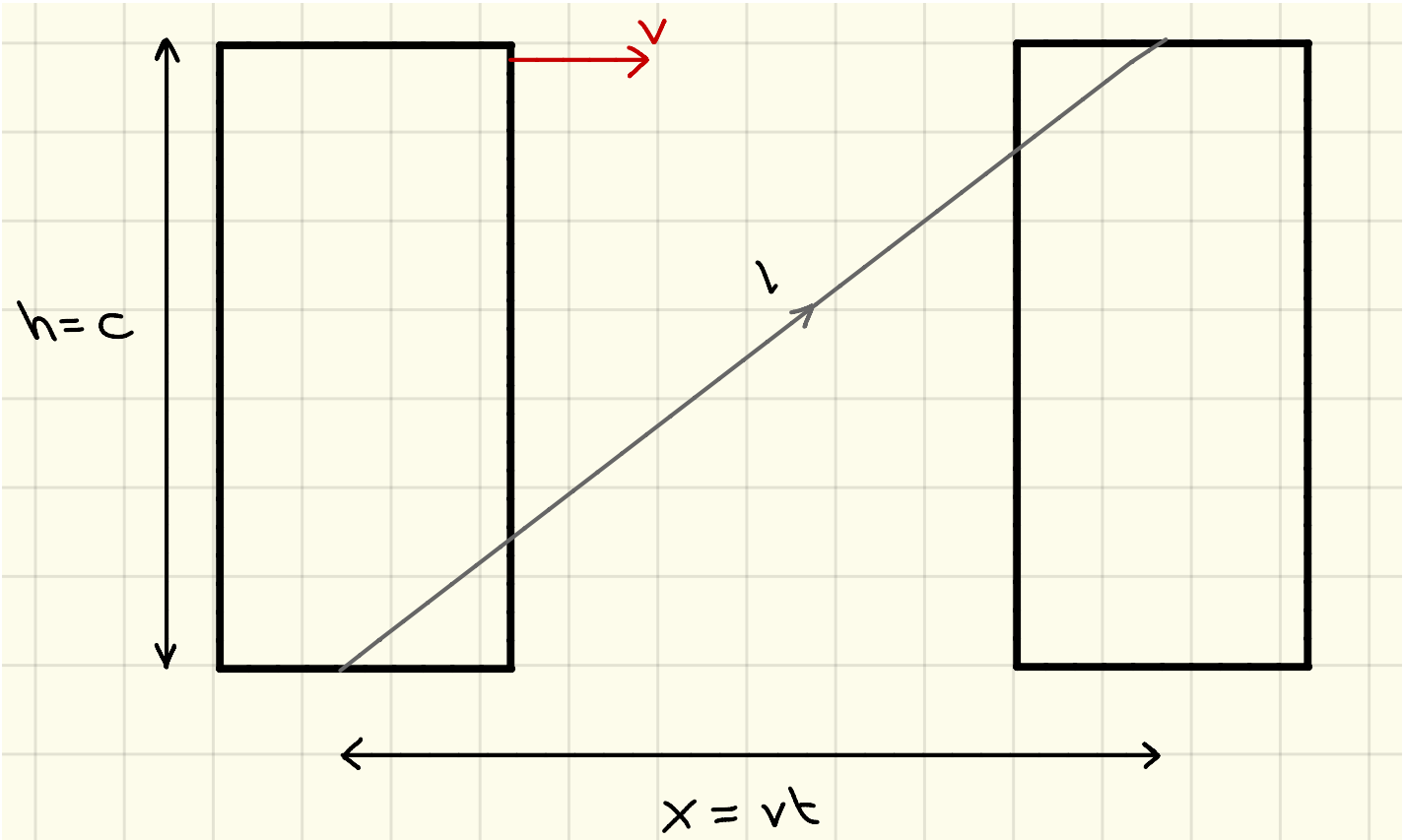
In 1905 Einstein posited that the speed of light c is constant in all reference frames. Consider a photon clock:



A photon bounces up and down and takes 1 second to go from the bottom to the top (this means the clock is 186,282 miles tall!).

Time Dilation

If the clock is moved at speed v horizontally:



By Pythagorus' theorem the photon now moves a distance

$$l = \sqrt{(vt)^2 + c^2}$$

Time Dilation

$$l = \sqrt{(vt)^2 + c^2}$$

As c is constant, the photon travels the distance l in time

$$t = \frac{l}{c} \implies l = ct.$$

Hence,

$$c^2t^2 = v^2t^2 + c^2.$$

Rearranging we have

$$t = \sqrt{\frac{c^2}{c^2 - v^2}} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \equiv \gamma$$

So one second on the stationary clock corresponds to $t = \gamma > 1$ (if $0 < v < c$) seconds as read off the moving clock.

Questions

The twins paradox: Consider two identical twins of exactly the same age. One twin stays on Earth while the second twin goes on a rocket to a distant planet. As the rocket departs both twins see each other in a moving frame and both twins think each other's clocks are running slow. When the rocket returns to Earth which twin is now the elder of the pair?