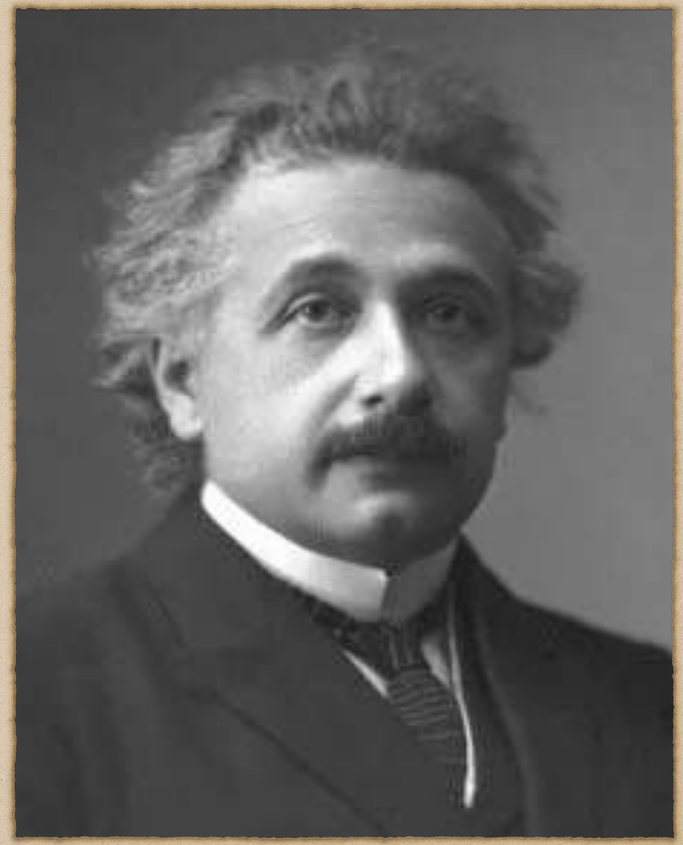
Some Paradoxes in Special Relativity

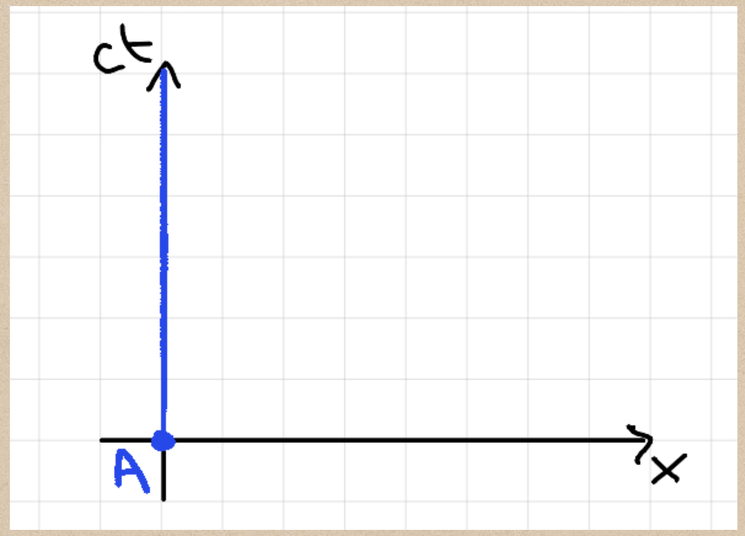
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1. Special Relativity

1. The laws of physics are independent of the inertial reference frame of the observer.
2. The speed of light in a vacuum is the same constant for all observers.



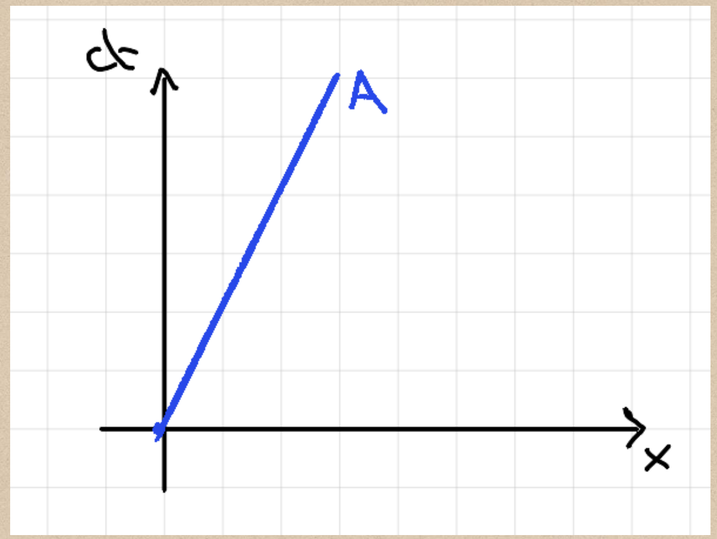
2. Space-time diagrams



A stays still at the coordinate x=0.

Their "world-line" coincides with the time (ct) axis.

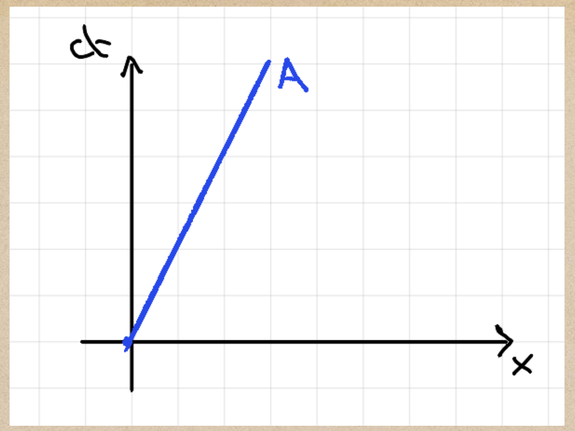
Suppose A moves at constant speed v in the x direction:



A's world-line is a straight line of the form $ct = \alpha x + \beta$

If at t=0, A is at x=0 then $\beta = 0$ and so $ct = \alpha x$

As A moves at v=x/t, then we find $\alpha = \frac{c}{v}$ giving $ct = \frac{c}{v}x$

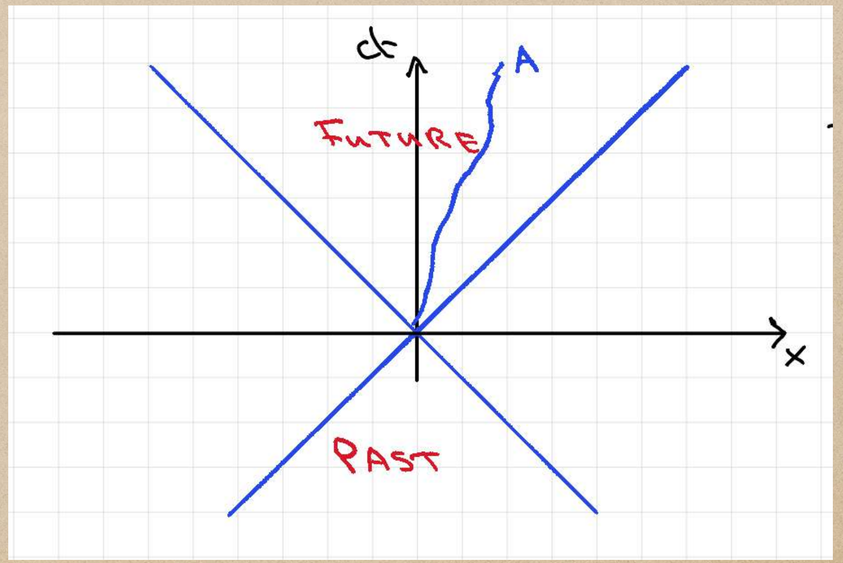


The gradient is $\alpha = \frac{c}{v}$

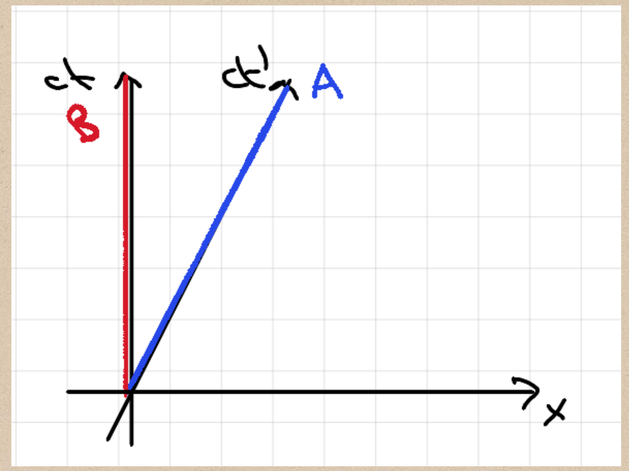
If $v = \pm c$ then the gradient is ± 1

If $0 \leq |v| < c$ then $1 \leq |\text{gradient}| < \infty$

(Relativistic mechanics prohibits objects from having a speed v which crosses the Speed of light, c.)



Suppose B is stationary at x=0, and watches A move at speed v=x/t:



A has their own frame of reference with coordinates (ct',x').

B deduces that the axis (x'=0) the ct' axis of A is the line ct=(c/v)x

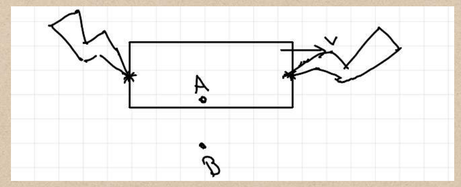
But, B, wonders what corresponds to the (t'=0) axis for A:

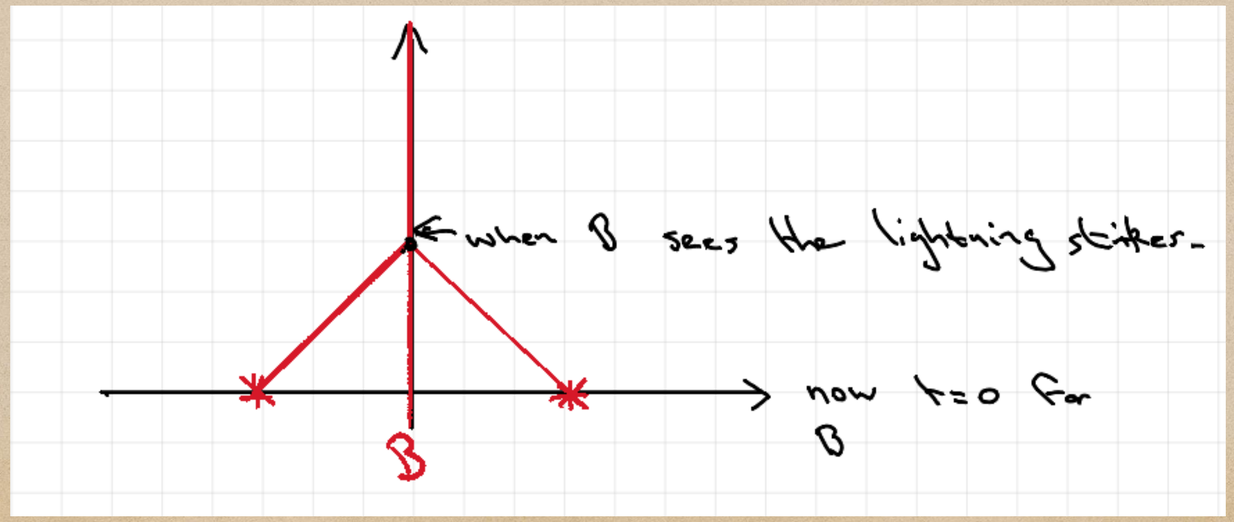
'when is now?'

3. When is now?

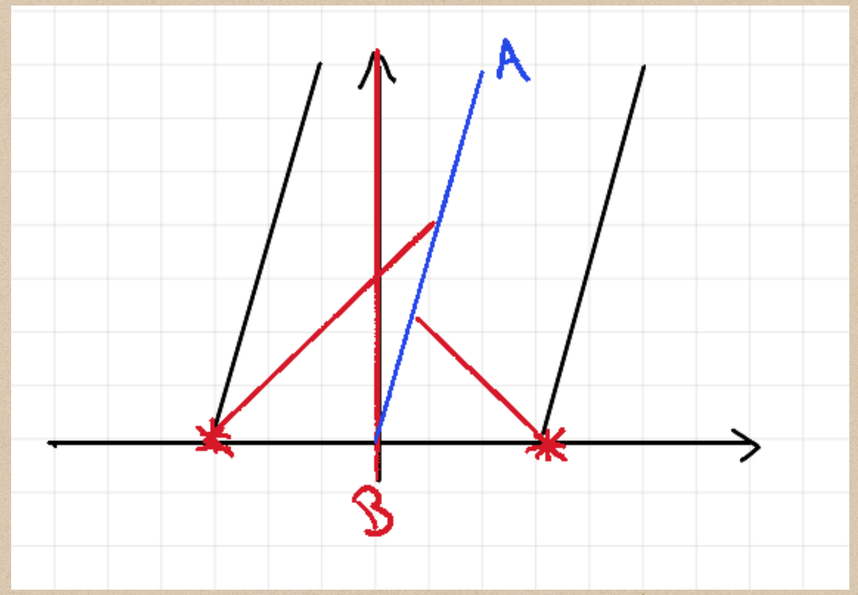
Einstein's thought (gedanken) experiment:

A sits in the middle of a train moving in a straight line at constant speed v=x/t relative to B who is stationary on a platform. As A passes B, lightning strikes the front and the back of the train that B simultaneously according to B:





A sees the front lightning strike hit first, and the rear lightning strike later:

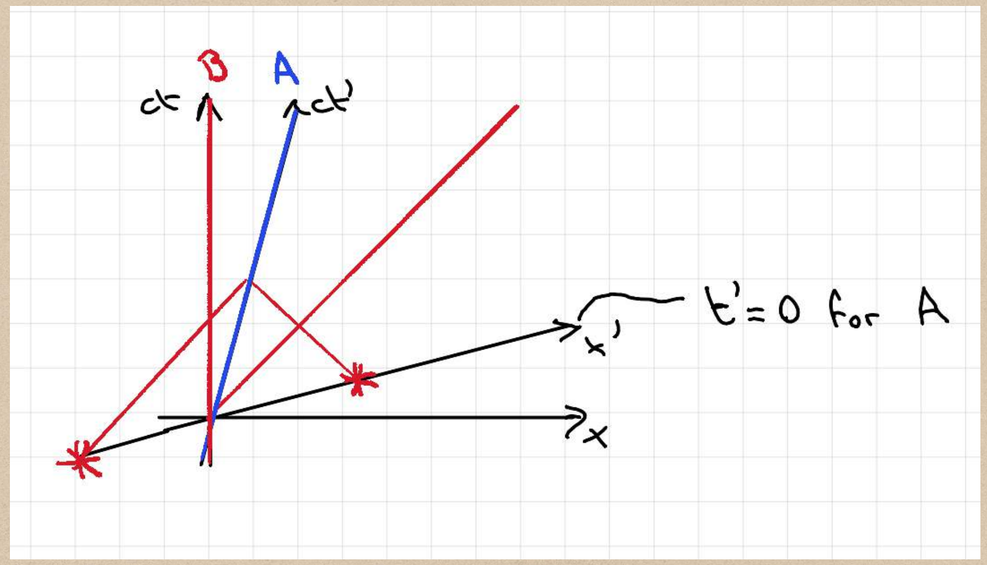


(Because A is moving towards the flash from the front of the train, and away from the flash at the rear.)

Two simultaneous events at different places happen at the same time.

We may ask when two events occur so that A witnesses them to be simultaneous.

For the lightning flashes to be simultaneous for A, the one at the rear would have to occur before the one at the front of the train:

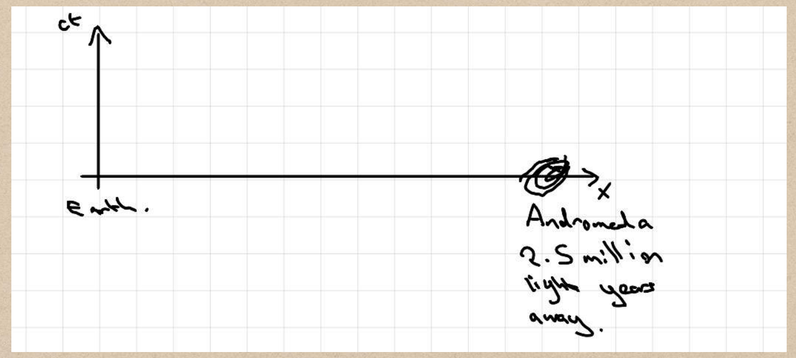


The line of simultaneity for A (at t'=0) defines the x' axis of A's reference frame (as seen by B).

It is a straight line tilted towards A's motion, flashes on that line equidistant from A are seen at the same time by A (but not by B).

The Andromeda Paradox

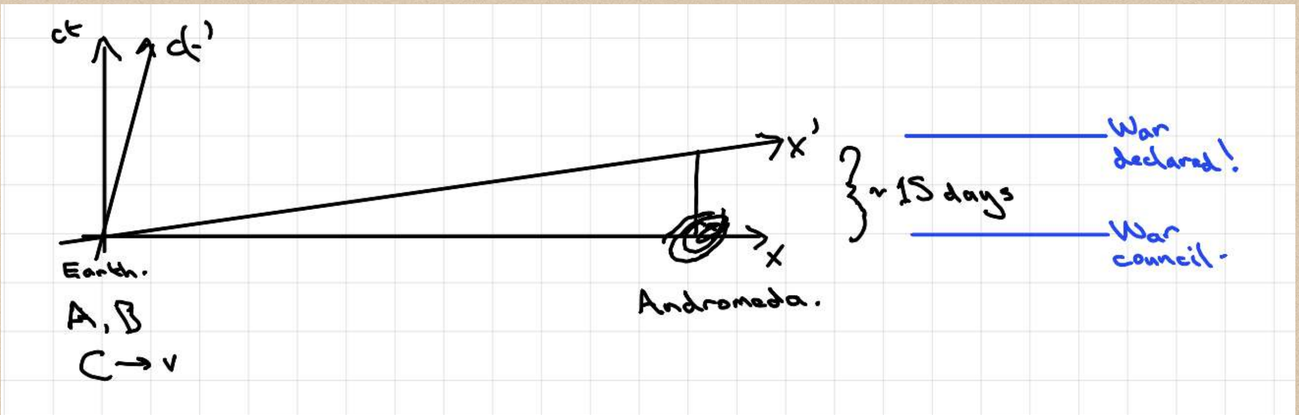
A and B stand still on Earth and look at the Andromeda galaxy (2.5 million light years away) and marvel that they are seeing light emitted 2.5 million years ago.



They speculate that right now a war council could be meeting on Andromeda to decide whether or not to attack Earth.

C walks past them at v=5 metres per second and disagrees.

C says "now" on Andromeda the war council is over and the fleet has already left to attack Earth.

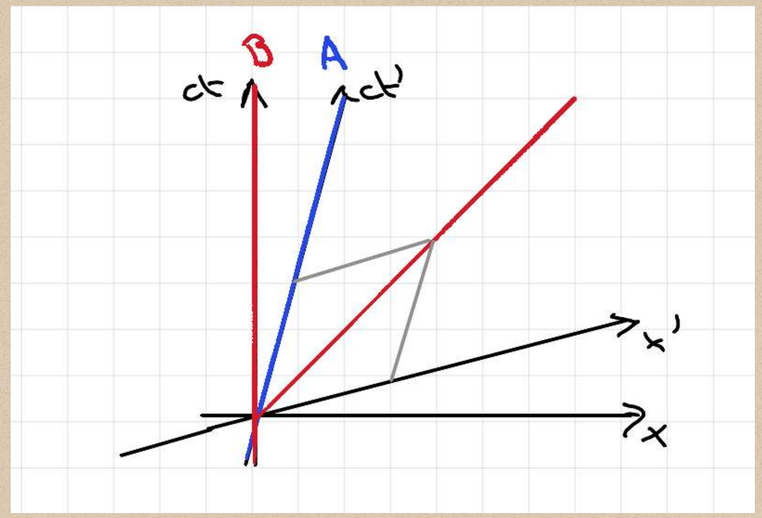


For C, their plane of simultaneity is tilted towards their motion, they disagree about when now is in Andromeda.

If C was to turn around and walk at v=-5 metres per second past A and B, they could argue that, now on Andromeda, no war council has been convened.

Observers in moving frames disagree about which events are simultaneous.

A and B agree on the speed of light.



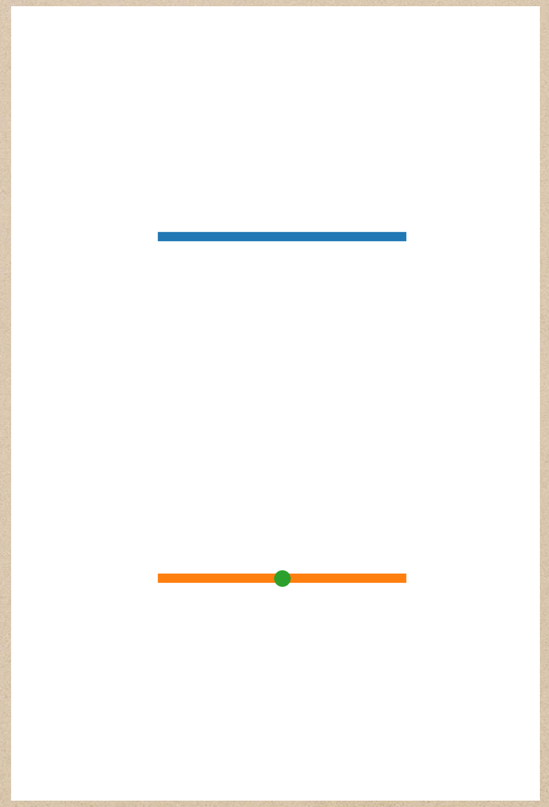
The equation for the x' axis is $ct = \left(\frac{v}{c}\right) x$

For x=2.5 million light years, and 5 metres per second we find

$ct \approx 5\text{ms}^{-1} \times (2.5 \times 10^6 \text{years})$, or $t \approx 15.2$ days.

4. Time Dilation

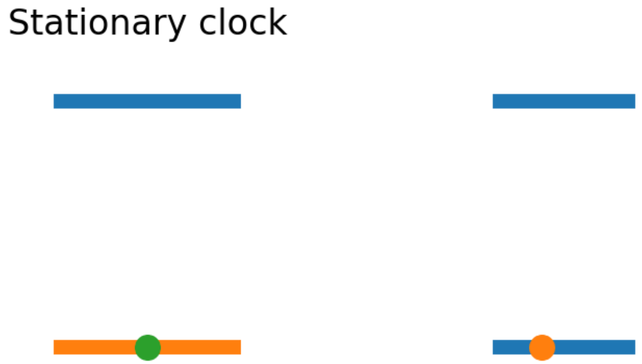
A photon clock “ticks” once each second, when the photon completes its round trip:

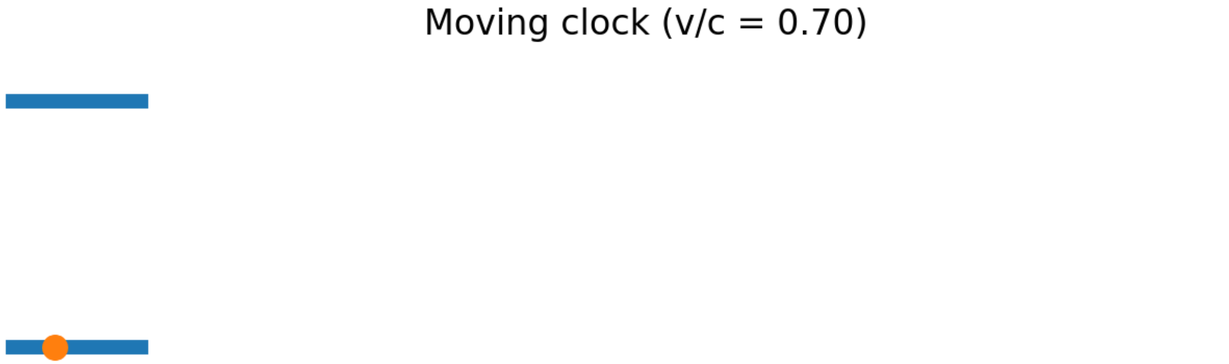


A moving photon clock:



Both clocks “ticking” together:



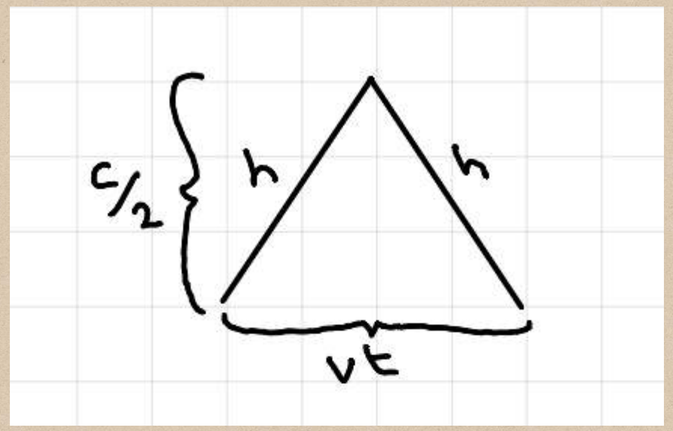


How much slower is the moving clock?

Suppose B is stationary, and A is moving at v=x/t and B observes A's clock.

They see the photon move at c along the hypotenuse, h, of a right angled triangle.

Suppose they see that t seconds have passed on their stationary clock when just one second has passed on A's clock:



The total distance travelled by the photon in A's clock is $L = 2h = 2\sqrt{(\frac{1}{2}vt)^2 + (\frac{c}{2})^2} = \sqrt{v^2t^2 + c^2}$

But the speed of light (the photon) is constant c=L/t and so

$$(c^2 - v^2)t^2 = c^2 \qquad \text{or} \qquad t = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \equiv \gamma(v)$$

For $0 \leq v < c$ then $1 \leq \gamma(v) < \infty$. Over t' seconds of A's clock, B measures $t = \gamma(v)t'$ (TIME DILATION).

The Twin Paradox

A and B are identical twins having the same age (in years).

A travels at v=4c/5 to a galaxy a distance d=4light years away. Then returns immediately at the same speed.

Both "see" each others time running slow:

$$\gamma(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$= \frac{1}{\sqrt{1 - \frac{16}{25}}}$$

$$= \frac{1}{\sqrt{\frac{9}{25}}}$$

$$= \frac{5}{3}$$

B computes that each leg of A's journey takes t=d/v= 5 years. B ages 10 years waiting for A to return.

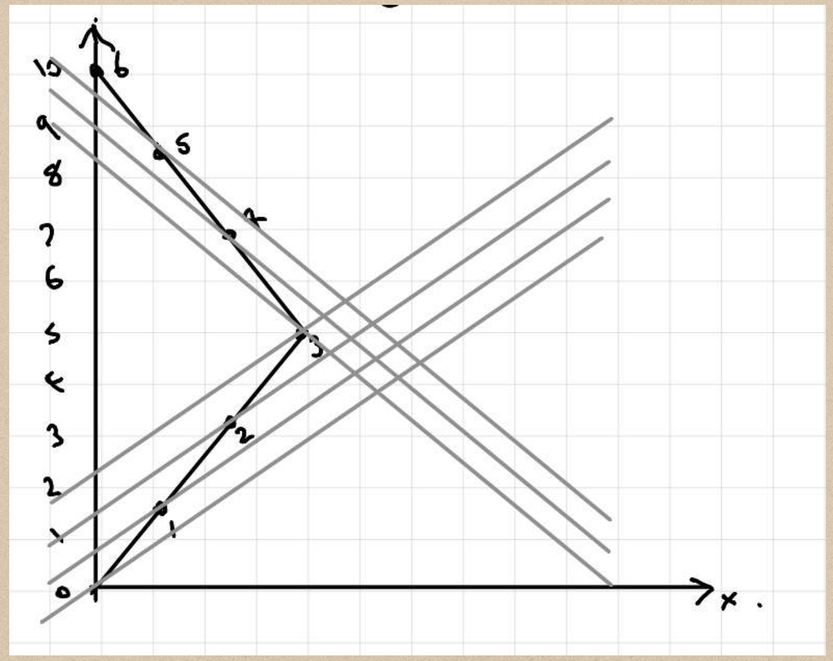
Due to time dilation, B knows that A's clock will run slow by a factor of
$\gamma(v) = \gamma(4c/5) = 3/5$.
So 5 years for B corresponds to 3 years for A.
B then expects A to age by just 6 years by the time they return (while B is 10 years older).

However, A makes the same argument and thinks that B is moving and that their clock runs slow.
A also expects to be the older twin when they meet again on Earth.

Who is right?

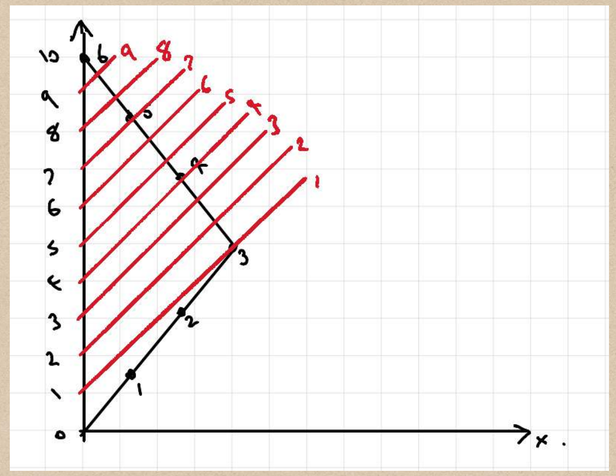
A is older: the journey they make involves turning around - the acceleration differentiates the two frames of reference.

We can see this by sketching the space-time diagram, and showing the planes of simultaneity for A on their journey.

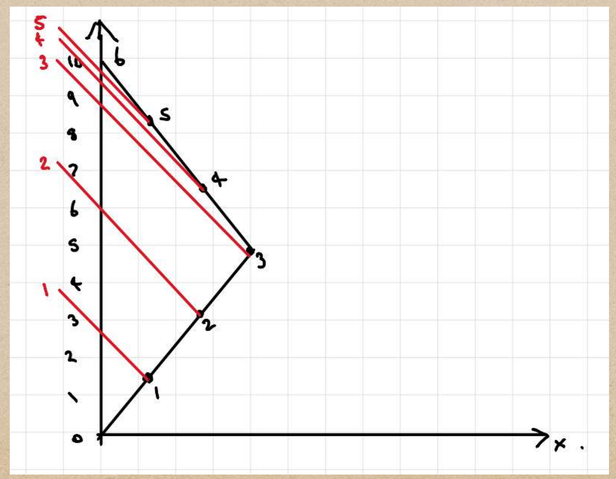


When A turns around the plane of simultaneity jumps suddenly by many years on Earth.

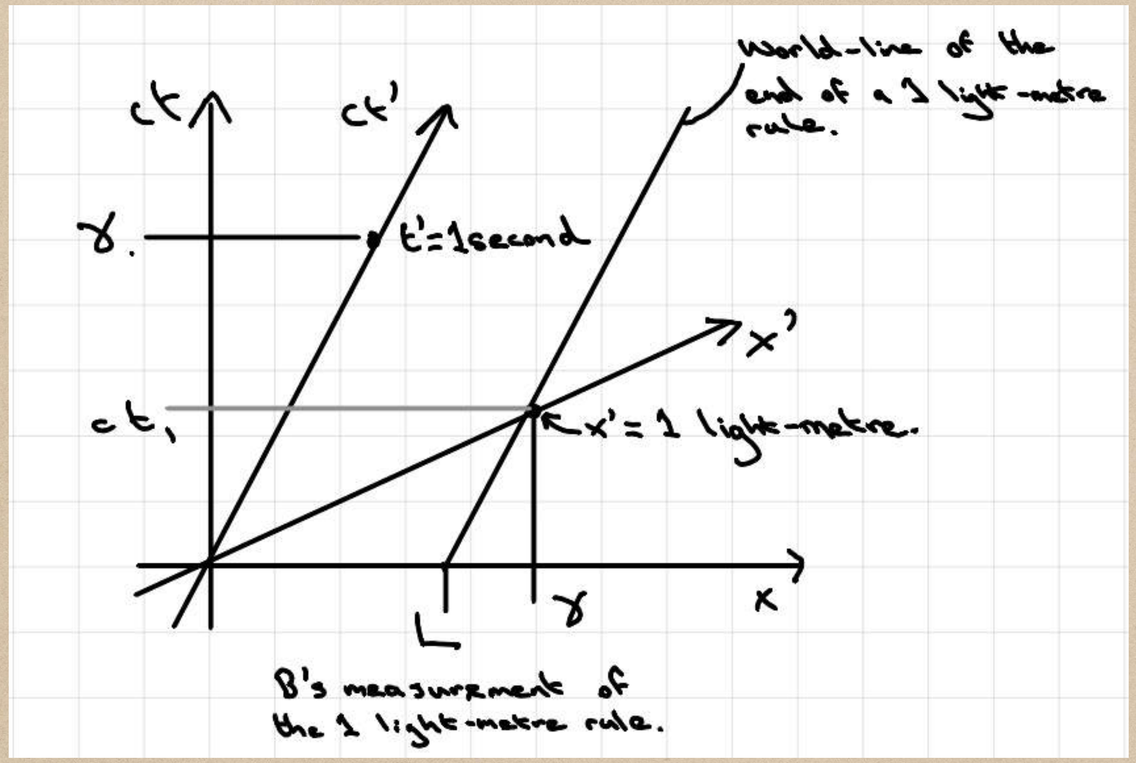
We could plot regular light signals (sent on birthdays) from B to A on a space-time diagram:



Or, instead, from A back to B on Earth:



5. Length Contraction



$$L + vt_1 = \gamma$$

The equation of A's x' axis is: $ct = \left(\frac{v}{c}\right)x$

and (ct_1, γ) lies on the x' axis, so: $ct_1 = \left(\frac{v}{c}\right)\gamma$

hence we have:

$$L + \frac{v^2}{c^2}\gamma = \gamma$$

$$\Longrightarrow\ L = (1 - \frac{v^2}{c^2})\gamma = \frac{\gamma}{\gamma^2} = \frac{1}{\gamma} \leq 1$$

If A measures a length L' then B measures it to be: $L = L'/\gamma \leq L'$

LENGTH CONTRACTION

The Ladder-Barn Paradox

Caleb has a 20 metre ladder and 10 metre barn.

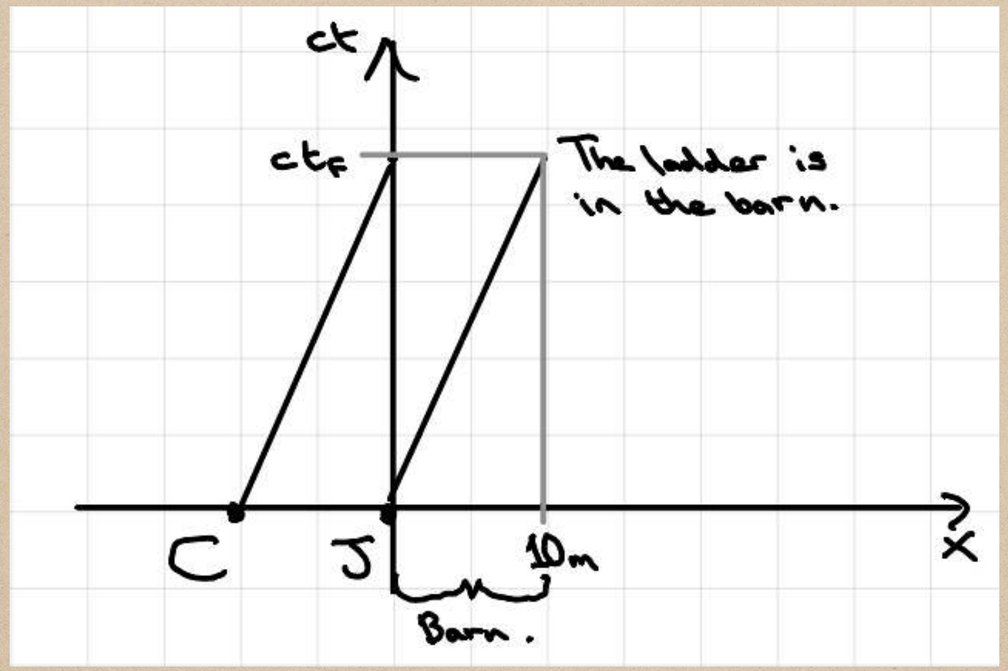
Caleb runs with the ladder at $v = \frac{\sqrt{3}}{2}c \implies \gamma(v) = \frac{1}{\sqrt{1-\frac{3}{4}}} = 2.$

Jeremy watches by the barn door and sees that the length of the ladder is contracted to $20/\gamma = 10$ metres and so it fits in the bran and Jeremy can shut the barn door once the ladder is completely in the barn

However, Caleb sees the ladder to be 20 metres, while the barn, which is moving towards him is shortened to $10/\gamma = 5$ metres.

How can they both be right?

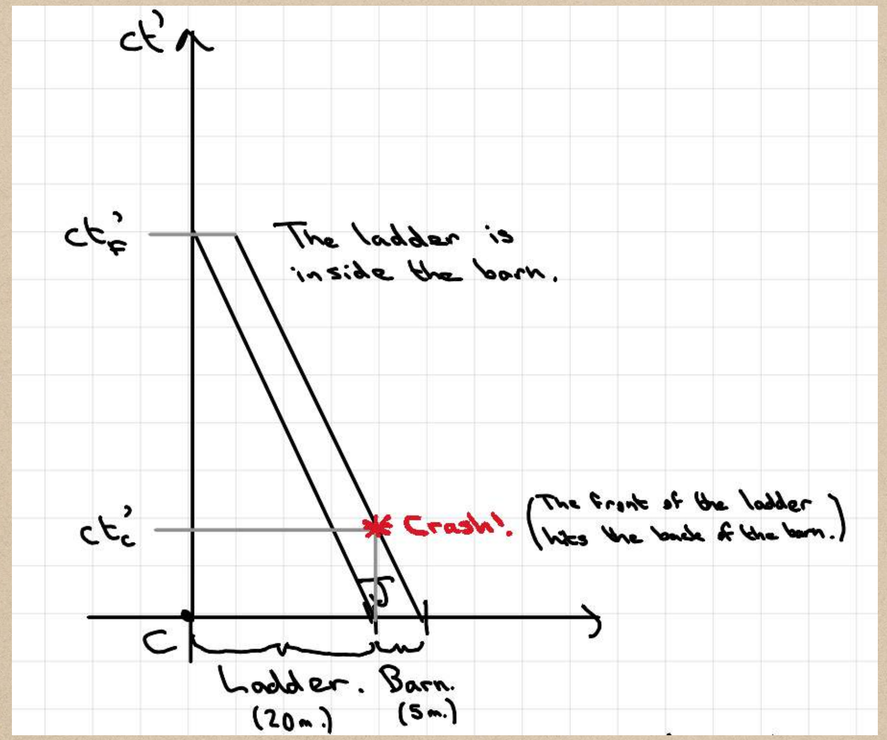
Jeremy's rest-frame:



$$t_F = d/v = 10/(\sqrt{3}c/2)$$

$$\implies ct_F = 20/\sqrt{3} \text{ light seconds}$$

Caleb's rest-frame:



$$t'_F = d/v = 20/(\sqrt{3}c/2)$$

$$\implies ct'_F = 40/\sqrt{3} \text{ light seconds}$$

$$t'_C = d/v = 5/(\sqrt{3}c/2)$$

$$\implies ct'_C = 10/\sqrt{3} \text{ light seconds}$$

Now $ct'_F - ct'_C = 30/\sqrt{3} \approx 17.3$ light seconds (this is the time the end of the ladder travels after the crash).

Suppose information about the crash travels at the speed of light (from the front to the back of the ladder) It would take $t' = 20/c$ seconds, or ct'=20 light seconds, by which time the end of the ladder is in the barn. Caleb and Jeremy both agree the barn door can be shut on the ladder.

Thanks for thinking and listening.

Further reading: Special Relativity for the Enthusiastic Beginner by David Morin

